# Tensor Algebra: A Combinatorial Approach to the Projective Geometry of Figures

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# Tensor Algebra: A Combinatorial Approach to the Projective Geometry of Figures

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**Abstract.** This paper explores the combinatorial aspects of symmetric and anti-symmetric forms represented in tensor algebra. The development of geometric perspective gained from tensor algebra has resulted in the discovery of a novel projection operator for the Chow form of a curve in $\mathbb{P}^3$ with applications to computer vision.

## 1 Introduction

The notation used in this paper is adapted from [16] with the novel addition of the symmetric operators which we will use to derive the representations for curves, surfaces and other non-linear algebraic maps. Also, we will maintain the use of vectorizations of the (anti)symmetric tensor forms, that offer an equivalent expression of the algebra as a coefficient ring defined over the field of real numbers ($\mathbb{R}$). This in turn can be expressed as the elements of a vector space on a computer. This approach to geometry is analytically equivalent to the approach taken by other authors studying geometry of linear objects [17], invariants [15, 8], multiple view geometry [16, 10] and also in the theory symmetric funtions [14]. The major contribution of this paper is the projection operator for the Chow form of a curve in space. This projection operator has allowed for a new class of curve based approaches to mulitple view geometry.

## 2 Tensor Basics

Tensors are a generalization of the concept of vectors and matrices. In fact vectors and matrices are 1 and 2-dimensional instances of a tensor. Tensors make it easier to understand the interaction of algebraic expressions that involve some type of mulitlinearity in their coefficients. The following sections will briefly introduce some of the algebra underlying the types of tensors we are most interested in.

### 2.1 Vector Spaces

Tensors are composed entirely from vector spaces. Vector spaces can be combined using a range of standard operators resulting in differently structured tensors. We will limit our study of the geometry herein to projective vector space $\mathbb{P}^n$. An element of an $n$-dimensional projective vector space in the tensor notation is denoted as $\mathbf{x}^{{}_mA_i^s} \in \mathbb{P}^n$. The symbol ${}_mA_i^s$ is called an indeterminant and identifies several important properties of the vector space. Firstly in order to better understand the notation we must rewrite $\mathbf{x}^A$ in the standard vector form. This is achieved by listing the elements of the vector space using the indeterminant as the variables of the expression. In this manor the symbol that adjoins the indeterminant is merely cosmetic, for example the equivalent vector space is, $\mathbf{x}^{{}_mA_i^s} \equiv [{}_mA_0^s, {}_mA_1^s, \ldots, {}_mA_n^s]^\top$, where $m$ identifies the multilinearity of the indeterminant, $s$ depicts the degree (or step) of the indeterminant. We show in the next section that there are several different types of degree that we will be concerned with and that these are used to denote a vectorization of the tensor form. The last element specifying the indeterminant is $i$, this a *choice* of the positioning of the elements in the vector, we most commonly refer to $i$ as the index of the indeterminant. The standard indexing is $i \in \{0 \ldots n\}$ for an $n$-dimensional projective vector space.

Indeterminants of a regular vector (vertical) space ($\mathbb{P}^n$) are called *contravariant* and indeterminants of a dual (horizontal) vector space in ${}^*\mathbb{P}^n$ (covector) are called *covariant*. The notation for a dual vector (covector) space is analogous to that for a regular vector space, $\mathbf{x}_{{}^*_mA_i^s} \equiv [{}^*_mA_0^s, {}^*_mA_1^s, \ldots, {}^*_mA_n^s]$, the only difference being that the vector is transposed. In the interests of compactness and clarity often we will abandon the entire set of labels for an indeterminant via an initial set of assignments. If this is the case assume that $i$ is any arbitrary scalar between $0$ and $n$ and $s, m = 1$. If an indeterminant is used in a covariant expression then the $*$ may also be omitted.

### 2.2 Tensors and Contraction

Tensor contraction is the process of eliminating vector spaces from a given tensor. Tensor contraction is achieved via a dot product of elements from a regular and dual vector space, resulting in a cancelation of both indeterminants.

$$\mathbf{x}_{{}^*_mA_i^s}\mathbf{x}^{{}_mA_i^s} \equiv [{}^*_mA_0^s, {}^*_mA_1^s, \ldots, {}^*_mA_n^s] \begin{bmatrix} {}_mA_0^s \\ {}_mA_1^s \\ \vdots \\ {}_mA_n^s \end{bmatrix} = \alpha \tag{1}$$

Since our vector spaces are projective the contraction results in the scalar $\alpha$. Most often we will be concerned with algebraically exact contractions that result in the scalar $0$, such a contraction is referred to as an incidence relation. Geometrically this usually corresponds to an exact point-hyperplane pair. The rules for tensor contraction are as follows;

- Contraction may only occur between common indeterminants, $\mathbf{x}_A\mathbf{y}^{AB} = \mathbf{z}^B$. Whereas is the case of $\mathbf{x}_A\mathbf{y}^{CB}$ no contraction can occur.
- Contraction occurs independant of the ordering of the indeterminants. For example $\mathbf{x}_{AB}\mathbf{y}^A = \mathbf{z}_B$ is equivalent to $\mathbf{y}^A\mathbf{x}_{BA} = \mathbf{z}_B$, this is called the Einstein summation notation.

## 3 Tensor Products

The basic tools used to construct the algebraic/geometric entities in the tensor notation are called operators. There are three different types of operators that we will use in this paper and for each operator we will maintain two differing representations, that of a tensor form and its equivalent vector form (Table 1). In Table 1 the symbols $\nu_n^d = \binom{d+n}{d} - 1$, $\eta_n^k = \binom{n+1}{k} - 1$ and $\pi_n^d = \prod_{i=1}^d n_i$.

The two different forms of the tensor are representative of the fact that we can always rewrite any tensor expression as an ordered vector of its coefficients. We call this alternative to the tensor form the vector form. Writing the tensor as a vector of coefficients abandons any symmetry present in the tensor this results in a less fruitful representation since it limits the way in which an equation can be contracted for symbolic derivations but in turn reduces the redundancy associated with the symmetry resulting in a more efficient representation for mappings between vector spaces.

| Operator | Symbol | Tensor Form | Vector Form |
|---|---|---|---|
| Segre | - | $\mathbf{x}^{A_i \ldots B_j}$ | $\mathbf{x}^{\alpha^d} \in \mathbb{P}^{\pi_n^d}$ where $\mathbf{x}^{A_i} \in \mathbb{P}^{n_A}$ |
| Antisymmetric (Step-$k$) | $[\ldots]$ | $\mathbf{x}^{[A_i \ldots B_j]}$ | $\mathbf{x}^{\alpha^{[k]}} \in \mathbb{P}^{\eta_n^k}$ |
| Symmteric (Degree-$d$) | $(\ldots)$ | $\mathbf{x}^{(A_i \ldots A_j)}$ | $\mathbf{x}^{\alpha^{(d)}} \in \mathbb{P}^{\nu_n^d}$ |

**Table 1.** Tensor Operators

### 3.1 Segre Operator

The Segre or *tensor product* operator is the most familiar of operators as it is generalization of the outer product rule for multiplication from linear algebra. A simple example of the Segre product operator is in the outer product multiplication of two vectors $\mathbf{x}^A \in \mathbb{P}^n$ and $\mathbf{y}^B \in \mathbb{P}^m$ the resulting tensor form is a purely contravariant matrix.

$$\mathbf{x}^A \mathbf{y}^B \equiv \begin{bmatrix} A_0B_0 & A_0B_1 & \cdots & A_0B_m \\ A_1B_0 & A_1B_1 & \cdots & A_1B_m \\ \vdots & \vdots & \ddots & \vdots \\ A_nB_0 & A_nB_1 & \cdots & A_nB_m \end{bmatrix} \equiv \mathbf{z}^{\alpha^2} \in \mathbb{P}^{nm} \tag{2}$$

If the multiplication had occured in the order $\mathbf{y}^B\mathbf{x}^A$ the then resulting matrix $\mathbf{z}^{BA}$ will simply be $\mathbf{z}^{AB}$ transposed. There is no analogue of the transpose from linear algebra in the tensor notation. Insted the equivalent of a transpose operation is just a shuffling of the indeterminants in the symbolic expression.

The equivalent vector form of the Segre product example given above is found by listing the elements of the resulting matrix $\mathbf{z}^{AB}$ in a vector ordered by the first indeterminant $A$, $\mathbf{x}^A\mathbf{y}^B \equiv \mathbf{z}^{\alpha^2} \in \mathbb{P}^{nm} = \begin{bmatrix} A_0B_0 & A_0B_1 & \cdots & A_nB_m \end{bmatrix}^\top$.

### 3.2 Antisymmetric Operator

The next operator of interest is the anti-symmetric operator ($[\ldots]$), the number of times the operator is applied to the vector space ($k$) is referred to as the step. Antisymmetrization in tensor algebra can be summarized as the process of multiplying tensor spaces according to following rules;

- Antisymmetrization may only occur between projective vector spaces of equivalent dimension, ie. for $\mathbf{x}^{[A}\mathbf{y}^{B]}$ to be admisable then given $\mathbf{x}^A \in \mathbb{P}^n$ and $\mathbf{y}^B \in \mathbb{P}^m$ then $m = n$.
- Labelling the result of $\mathbf{x}^{[A}\mathbf{y}^{B]}$ as $\mathbf{z}^{[AB]}$ where $\mathbf{x}^A, \mathbf{y}^B \in \mathbb{P}^n$. Denoting the indeterminants of $\mathbb{P}^n$ as belonging to the set $\alpha_i \in \{0 \ldots n\}$ then the following rules apply to the indeterminants $A_{\alpha_i} B_{\alpha_j}$,
  1. if $\alpha_i = \alpha_j$ then $[A_{\alpha_i} B_{\alpha_j}] = 0$
  2. if $\alpha_i \neq \alpha_j$ then $[A_{\alpha_i} B_{\alpha_j}] = \frac{1}{p!}\mathrm{sign}(\beta\alpha_i\alpha_j)A_{\alpha_i} B_{\alpha_j}$ where $\beta = \alpha/\{\alpha_i, \alpha_j\}$, which is the entire set of indeterminants modulo the ones contained in the expression $(\alpha_i, \alpha_j)$, also $p$ is the number of different ways you can reorder $A_{\alpha_i} B_{\alpha_j}$.
- The antisymmetrization of step $n+1$ or greater of elements from $\mathbb{P}^n$ will be $0$ for projective vector spaces ($\mathbf{x}^{[A_0 \cdots}\mathbf{y}^{A_n \cdots}\mathbf{z}^{B_j]} = 0$).

Now we apply these rules to a simple example,

$$\mathbf{x}^{[A}\mathbf{y}^{B]} \equiv \begin{bmatrix} 0 & [A_0B_1] & [A_0B_2] \\ [A_1B_0] & 0 & [A_1B_2] \\ [A_2B_0] & [A_2B_1] & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{A_0B_1}{2} & -\frac{A_0B_2}{2} \\ -\frac{A_1B_0}{2} & 0 & \frac{A_1B_2}{2} \\ \frac{A_2B_0}{2} & -\frac{A_2B_1}{2} & 0 \end{bmatrix} \equiv \mathbf{z}^{[AB]} \in \mathbb{P}^{\eta_2^2} \tag{3}$$

Analyzing the sign of the elements in the tensor in a little more detail we see that $[A_0B_1] = A_0B_1$ since $(201)$ is an even permutation of $(012)$ resulting in positive sign. Also $[A_0B_2] = -A_0B_2$ since $(102)$ is an odd permutation of $(012)$. Another point to note about the tensor given in the example above is that assuming the field for the tensor operations is commutative their are only 3 unique elements involved in its construction. These elements are repeated in an antisymmetric fashion across the main diagonal.

This is due to the fact that $k$ anitsymmetrizations of a $n$-dimensional projective vector space contains $\eta_n^k + 1$ indeterminates (from Table 1). This is precisely equivalent to the number of ways that we can reorder the members of $\alpha$ in a *strictly increasing* manor. For example the unique elements and associated vector spaces of $\mathbb{P}^{\eta_2^2}$ and $\mathbb{P}^{\eta_3^3}$ are; $[+01, -02, +12]$ and $[-012, +013, -023, +123]$ Here we have ordered the elements of these sets in a lexicographic order.

### 3.3 Symmetric Operator

The next operator of interest is the symmetrization operator. The symmetrization operator allows us to create symmetric expansions of the vector space at hand, the number of symmetrizations applied is referred to the degree $d$ of operator. The vectorized version of the symmetrization operator is known in the literature as the Veronese embedding [9]. Symmetrization may also be summarized according to the following set of rules;

– Symmetrization may only occur between projective vector spaces of equivalent dimension, ie. for $\mathbf{x}^{(A}\mathbf{y}^{B)}$ to be admisable then given $\mathbf{x}^A \in \mathbb{P}^n$ and $\mathbf{y}^B \in \mathbb{P}^m$ then $m = n$. As a matter of convention we will usually denote a symmetrization with the same indeterminant repeated ie. $\mathbf{x}^{(A}\mathbf{y}^{A)}$.
– Labelling the result of $\mathbf{x}^{(A} \cdots \mathbf{y}^{A)}$ as $\mathbf{z}^{(A \cdots A)}$ where $\mathbf{x}^A \in \mathbb{P}^n$. Denoting the indeterminants of $\mathbb{P}^n$ as belonging to the set $\alpha = \{0 \ldots n\}$ then the following rule applies to the indeterminants $A_{\alpha_i} \cdots A_{\alpha_j}$,
  - for all $\alpha_i \ldots \alpha_j$, $(A_{\alpha_i} \cdots A_{\alpha_j}) = \mathrm{perms}(A_{\alpha_i} \cdots A_{\alpha_j})$
– $(A_{\alpha_i} B_{\alpha_j}) = \frac{1}{p!} A_{\alpha_i} B_{\alpha_j}$ where $p$ is the number of unique permutations possible by reordering $A_{\alpha_i} B_{\alpha_j}$.

These rules state that the symmetrization produces indeterminants that are equal for every possible reordering of the indexes, thus enabling the symmetry. Applying this to a simple example we have,

$$\mathbf{x}^{(A}\mathbf{y}^{A)} \equiv \begin{bmatrix} (A_0 A_0) & (A_0 A_1) & (A_0 A_2) \\ (A_1 A_0) & (A_1 A_1) & (A_1 B_2) \\ (A_2 A_0) & (A_2 B_1) & (A_2 A_2) \end{bmatrix} = \begin{bmatrix} A_0 A_0 & \frac{A_0 A_1}{2} & \frac{A_0 A_2}{2} \\ \frac{A_0 A_1}{2} & A_1 A_1 & \frac{A_1 A_2}{2} \\ \frac{A_0 A_2}{2} & \frac{A_1 A_2}{2} & A_2 A_2 \end{bmatrix} \equiv \mathbf{z}^{(AB)} \in \mathbb{P}^{\nu_2^2} \tag{4}$$

Again assuming a commutative field we have only 6 ($\nu_2^2 + 1$ = 6, Table 1) unique combinations of indeterminants in the tensor given above. This is precisely equivalent to the number of ways that we can reorder the members of $\alpha$ in a *strictly non-decreasing* manor. For example the unique elements of $\nu_2^2$ and $\nu_1^3$ are; $[00, 01, 11, 02, 12, 22]$ and $[000, 001, 011, 111]$. Again we have ordered the elements of these sets in a lexicographic order.

# 4 Linear Features in $\mathbb{P}^2$ and $\mathbb{P}^3$

The application of the tensor operators given in Table 1 to vector spaces gives us a means represent the geometry of various features we encounter in computer vision as the embedding of the (codimension 1) coefficient ring of the feature into a vector space.

## 4.1 Geometry of the Antisymmetrization Operator

Linear features are defined as any feature the that can be expressed in terms of strictly linear coefficients, this is to say the highest degree of the monomials composing the geometric form is 1. In order to construct the total set of linear features in a projective vector space $\mathbb{P}^n$ we use a geometric interpretation of the antisymmetric operator defined in the previous section.

This ammounts to viewing the application of the antisymmetrization operator to a set of contravariant vectors as the join ($\wedge$) of two vectors (ie. $\mathbf{x}^{[A}\mathbf{y}^{B]} \equiv \mathbf{x}^A \wedge \mathbf{y}^B$). Likewise the application of the antisymmetrization operator to two or more covariant vectors is the meet ($\vee$) (ie. $\mathbf{x}_{[A}\mathbf{y}_{B]} \equiv \mathbf{x}_A \vee \mathbf{y}_B$). This equivalent interpretation of the antisymmetrization operator originates from the Grassmann-Cayley algebra [5, 17].

The Degrees Of Freedom (DOF) of contravariant hyperplanes is the the same as the size of the space they are embedded in, whereas covariant hyperplanes will always have 1 DOF. Generically contravariant vectors are points in a projective space and covariant vectors are lines, planes for $\mathbb{P}^2$, $\mathbb{P}^3$ respectively.

The interpretation of the join ($\wedge$) and meet ($\vee$) operators is quite literal as they denote the joining of two or more contravariant vectors (points) and the intersection of two or more covariant vectors (lines, planes, etc..).

In $\mathbb{P}^2$ or the projective plane the only linear features not including the plane itself are the point and line. Table 2 summarises the representation and the DOF for linear features in the projective plane ($[A_0, A_1, A_2] \in \mathbb{P}^2$). Similarly, Table 3 summarises the representation and the DOF for linear features in projective space ($[a_0, a_1, a_2, a_3] \in \mathbb{P}^3$).

| Feature | $\mathbb{P}^2$ | ${}^*\mathbb{P}^2$ | $\mathbf{DOF}_i$ | Embedding |
|---|---|---|---|---|
| Points | $\mathbf{x}^{A_0}$ | $\mathcal{A}_* : \mathbf{x}^{A_0} \rightarrow \epsilon_{A_0A_1A_2}\mathbf{x}^{A_0} = \mathbf{x}_{[A_1A_2]}$ | 2 | $\mathbb{P}^2$ |
| Lines | $\mathbf{x}^{[A_0A_1]}$ | $\mathcal{A}_* : \mathbf{x}^{[A_0A_1]} \rightarrow \epsilon_{A_0A_1A_2}\mathbf{x}^{A_0A_1} = \mathbf{x}_{A_2}$ | 1 | $\mathbb{P}^2$ |

**Table 2.** Linear features and their duals in $\mathbb{P}^2$

| Feature | $\mathbb{P}^3$ | ${}^*\mathbb{P}^3$ | $\mathbf{DOF}_s$ | Embedding |
|---|---|---|---|---|
| Points | $\mathbf{x}^{a_0}$ | $\mathcal{A}_* : \mathbf{x}^{a_0} \rightarrow \epsilon_{a_0a_1a_2a_3}\mathbf{x}^{a_0} = x_{[a_1a_2a_3]}$ | 3 | $\mathbb{P}^3$ |
| Lines | $\mathbf{x}^{[a_0a_1]}$ | $\mathcal{A}_* : \mathbf{x}^{[a_0a_1]} \rightarrow \epsilon_{a_0a_1a_2a_3}\mathbf{x}^{a_0a_1} = x_{[a_2a_3]}$ | 4 | $\mathbb{P}^5$ |
| Planes | $\mathbf{x}^{[a_0a_1a_2]}$ | $\mathcal{A}_* : \mathbf{x}^{[a_0a_1a_2]} \rightarrow \epsilon_{a_0a_1a_2a_3}\mathbf{x}^{a_0a_1a_2} = x_{a_3}$ | 1 | $\mathbb{P}^3$ |

**Table 3.** Linear features and their duals in $\mathbb{P}^3$

Tables 2 and 3 also demonstrate the process of dualization for linear feature types via the dualization mapping ($\mathcal{A}_*$) [7]. Dual representations in the antisymmetric (Grassmann-Cayley) algebra are equivalent covariant forms of the same geometric object interchanging the position and structure indeterminants using an alternating contraction ($\epsilon_{\beta_0 \ldots \beta_n}$). In addition to this the dualization function ($\mathcal{A}_*$) in the antisymmetric algebra is commutative.

### 4.2 Lines in $\mathbb{P}^3$

The only feature included in the prior discussion that cannot be classified as a hyperplane is the variety of the line embedded in $\mathbb{P}^3$. From Table 3 we denote the line joining two points as $\mathbf{x}^{[a_0}\mathbf{y}^{a_1]} \simeq \mathbf{l}^{[a_0a_1]} \equiv \mathbf{l}^{\omega} \in \mathbb{P}^5$ or dually as the intersection of two planes $\mathbf{x}_{[a_0}\mathbf{y}_{a_1]} \simeq \mathbf{l}_{[a_0a_1]} \equiv \mathbf{l}_{\omega} \in {}^*\mathbb{P}^5$. Therefore, analyzing the $6$ coefficients of the line,

$$\mathbf{l}^{[a_ia_j]} \equiv \mathbf{l}^{\omega} \in \{[a_0a_1], [a_0a_2], [a_1a_2], [a_0a_3], [a_1a_3], [a_2a_3]\} \tag{5}$$

we see that they are degree two monomials formed by taking the determinant of rows $i$ and $j$ (where $i < j$) of the the $[4 \times 2]$ matrix formed by the two contravariant points on the line. Similarly the dual representation of the line is formed by taking the determinant of columns $i$ and $j$ from the $[2 \times 4]$ matrix formed by the two covariant vectors depicting the planes that meet to form the line. This representation of the line is called the Plucker line.

The coefficients of the Plucker line have only 4 DOF (instead of 6). This is due to the loss of one DOF in the projective scaling and another due to a special relationship between the coefficients of the line called the quadratic Plucker relation. Writing the equations of the line in the skew-symmetric tensor form and taking the determinant of $\mathbf{l}^{[a_i a_j]}$ we find, $([a_0a_1][a_2a_3] - [a_0a_2][a_1a_3] + [a_0a_3][a_1a_2])$ which is the quadratic Plucker relation for the line. If the quadratic Plucker relation does not equal zero then the coefficient vector in $\mathbb{P}^5$ does not correspond to a line in $\mathbb{P}^3$.

## 5 Degree-$d$ Features in $\mathbb{P}^2$ and $\mathbb{P}^3$

The next group of features we are interested in expressing in tensor algebra are curves and surfaces in $\mathbb{P}^2$ and $\mathbb{P}^3$. These features are essentially non-linear as they are composed of monomials which are of degree $\geq 2$.

### 5.1 Hypersurfaces in $\mathbb{P}^2$ and $\mathbb{P}^3$

Curves in $\mathbb{P}^2$ and surfaces in $\mathbb{P}^3$ are both instances of codimension 1 hypersurfaces, which we will construct from the symmetric operator $(\ldots)$ as demonstrated in Table 4. In the Algebraic-Geometry literature hypersurfaces are referred to as the coefficient ring corresponding to the degree-$d$ Veronese embedding of a complex vector space $\mathbf{x}^\alpha \in \mathbb{CP}^n$ [9]. This means that hypersurfaces are generically points in a $\mathbb{CP}^{\nu_n^d}$ dimensional space, where $\nu_n^d = \binom{n+d}{n} - 1$, thus they have $\nu_n^d - 1$ DOF. This results in a degree-$d$ hypersurface $\mathbf{H}_{\alpha^{(d)}}$ (where $\mathbf{x}^\alpha \in \mathbb{P}^n$) satisfying the equation $\mathbf{H}_{\alpha^{(d)}}\mathbf{x}^{\alpha^{(d)}} = 0$, which is the incidence relation for hypersurfaces. Hypersurfaces allow us to catagorize implicit curves and surfaces into different classes according to their total degree (d) of the embedding.

| Hypersurface | Regular | Dual | **DOF** | Embedding |
|---|---|---|---|---|
| $\mathbb{CP}^2$ | $\mathbf{x}_{(A\ldots A)}$ | $\mathcal{S}_* : \mathbf{x}_{(A\ldots A)} \to \mathbf{x}^{(A\ldots A)}$ | $\nu_2^d$ | $\mathbb{CP}^{\nu_2^d}$ |
| $\mathbb{CP}^3$ | $\mathbf{x}_{(a\ldots a)}$ | $\mathcal{S}_* : \mathbf{x}_{(a\ldots a)} \to \mathbf{x}^{(a\ldots a)}$ | $\nu_3^d$ | $\mathbb{CP}^{\nu_3^d}$ |

**Table 4.** Degree-$d$ hypersurfaces and their duals in $\mathbb{P}^2$ & $\mathbb{P}^3$

**Curves in $\mathbb{P}^2$** A common example of an implicit hypersurface in $\mathbb{P}^2$ is the quadratic hypersurface (conic) We can write this as a vector of coefficients and use the incidence relation of hypersurfaces to get the typical equation for a point $\mathbf{x}^A$ on a degree 2 curve (conic).

$$\mathbf{c}_{(AA)}\mathbf{x}^A\mathbf{x}^A \equiv \mathbf{c}_{A^{(2)}}\mathbf{x}^{A^{(2)}} = 0 \quad (6)$$

All quadratic curves in the plane can be given in terms of this incidence relation by varying the coefficient vector $\mathbf{c}_{A^{(2)}}$ of the hypersurface. This concept can be used to define curves of any degree in $\mathbb{P}^2$ and likewise surfaces of any degree in $\mathbb{P}^3$. The dualization operator for symmetrically embedded hypersurfaces ($\mathcal{S}_*$) has a much more complicated action on the coefficient ring (see [6] for more details), however it simplifies in the case of degree 2 hypersurfaces to being simply the adjoint of the original symmetric matrix of coefficients (ie. $\mathcal{S}_*(\mathbf{c}_{(AA)}) \equiv \mathrm{adj}(\mathbf{c}_{(AA)}) = \mathbf{c}^{(AA)}$). Also in the degree 2 case the tangent cone at a single point $\mathbf{p}^\alpha$ to the surface or curve is identical since degree 2 hypersurfaces embedded in $\mathbb{P}^n$ (where $n \geq 2$) have no shape characteristics.

### 5.2 Hypersurfaces in $\mathbb{P}^5$ : Chow Forms

For practical purposes we wish to have a single equation (codimension 1 hypersurface) to define the locus of a curve in space. Arthur Cayley in the first of two papers [1, 2] on the topic describes the problem like this; *'The ordinary analytical representation of a curve in space by the equations of two surfaces passing through the curve is, even in the case where the curve is the complete intersection of the two surfaces, inappropriate as involving the consideration of surfaces which are extraneous to the curve'*.

The use of the *extraneous* surfaces in the representation of the curve can be abandoned if insted we consider the curve as the intersection of a cone of Plucker lines having as its apex a variable point in $\mathbf{p}^a \in \mathbb{P}^3$. In this manor the equation of the curve depicts the intersection of every cone of Plucker lines with apex $\mathbf{p}^a$ not incident with the curve. So for this purpose we represent the equation of a curve embedded in $\mathbb{P}^3$ as the degree-$d$ embedding of the Plucker line $\mathbf{x}^\omega \in \mathbb{P}^5$ (Table 3).

This results in a degree-$d$ curve satisfying the equation $\mathbf{C}_{\omega^{(d)}}\mathbf{x}^{\omega^{(d)}} = 0$ this type of equation is referred to as the *Chow Form* of the curve (see [6] 1-cycle) after its more contemporary definition by Chow and van der Waerden [3]. There exists $\nu_5^d = \binom{d+5}{d}$ coefficients in the ring of the Chow Form of a degree-$d$ curve in $\mathbb{P}^3$. Due to the redundancy of the Plucker relation for a line, the $\mathbf{DOF}$ of the Chow Form are [9, 12], $\mathbf{DOF}_{\mathrm{cf}} = \xi_5^d = \binom{d+5}{d} - \binom{d-2+5}{d-2} - 1$.

It is important to be able to characterise the Plucker relations that lead to the ancillary constraints on the Chow Form. Cayley [1, 2] was able to show that this condition for curves in space is equivalent to the following equation,

$$\frac{\partial^2 \mathbf{C}_{\omega^{(d)}}}{\partial \omega_0 \partial \omega_5} + \frac{\partial^2 \mathbf{C}_{\omega^{(d)}}}{\partial \omega_1 \partial \omega_4} + \frac{\partial^2 \mathbf{C}_{\omega^{(d)}}}{\partial \omega_2 \partial \omega_3} = 0 \tag{7}$$

which we can manufacture by symbolic differentiation of the Chow polynomial. This condition does suffice for $d \leq 3$ however for $d \geq 4$ further constraints constructed from higher order derivatives of the Chow form must be used.

A key property of the Chow form of the curve is its invariance to projective space transforms, this property is inherited from the underlying Plucker line representation. This makes the Chow form of the curve ideal for use in solving problems based in projective geometry (eg. those relating to projective observations) since the shape properties for all curves are invariant.

# 6 Projection Operators

The projection operator is an in injective projective transformation of vector space. Projection (in a geometric context) is the process projecting a feature to a lower dimensional embedding. The abstract definition of a projection is the mapping,

$$\pi_p : \mathbb{P}^n - \{p\} \to \mathbb{P}^{n-1} \tag{8}$$

which is projection from $\mathbb{P}^n$ to $\mathbb{P}^{n-1}$ from a point $p$ where $p \in \mathbb{P}^n$ but doesn't intersect $\mathbf{X}$ where $\mathbf{X}$ is the projective variety (feature) that is being projected. Primarily we will be interested in projections of features in $\mathbb{P}^3$ to $\mathbb{P}^2$.

## 6.1 Embedded Projection Operators

**Projection Operators for Surfaces** First we will develop the projection operator for surfaces in $\mathbb{P}^3$. In this setting the projection of a surface into the image is achievable by a projection of the dual tangent cone to the surface with the camera center as its apex ($\mathbf{S}^{a^{(d)}}$) to the dual plane curve in the image ($\mathbf{c}^{A^{(d)}}$). This projection utilizes the vectorization of the degree $d$ symmetric embedding of the Point-to-Point projection operator. However, since the degree of the dual of an algebraic surface is $d(d-1)^2$ in a practical setting only the projection of a degree 2 surface is reasonably attainable. Furthermore its seems to be an open question as to whether or not there exists a closed form manor of deriving the dual of surface with degree $> 2$. This operator has been noted and used by several authors [4, 11, 13] in the degree 2 case.

$$\mathbf{c}^{A^{(d)}} \simeq \mathbf{P}^{A^{(d)}}_{a^{(d)}} \mathbf{S}^{a^{(d)}} \tag{9}$$

**Projection Operators for Curves in $\mathbb{P}^3$** We now develop a novel projection operator for the Chow form of the curve in $\mathbb{P}^3$. We saw in section 5.2 that a curve in $\mathbb{P}^3$ is represented as the coefficient ring $\mathbf{c}_{\omega^{(d)}}$ where $\mathbf{x}^{\omega} \in \mathbb{P}^5$ is the coefficient ring of a Plucker line. Using the transpose of the Line-to-Point projection operator $\mathbf{P}^{[a_0 a_1]}_{A}$ [13] and rewriting it as $\mathbf{P}^{[a_0 a_1]}_{A} \equiv \mathbf{P}^{\omega}_{A}$, where $\mathbf{x}^{\omega} \in \mathbb{P}^5$ is the Plucker embedding of the line. We now have the basic linear mapping between a Plucker line passing through the camera center ($\mathbf{e}^{a}$) and the point at which it intersects the image plane. The vectorization of the symmetric degree $d$ embedding of the transpose of the Line-to-Point operator gives us the projection operator,

$$\mathbf{x}_{A^{(d)}} \simeq \mathbf{P}^{\omega^{(d)}}_{A^{(d)}} \mathbf{c}_{\omega^{(d)}} \tag{10}$$

which projects the coefficients of the Chow form of the curve from $\mathbb{P}^3$ to its image a hypersurface (or curve) in $\mathbb{P}^2$. This projection operation is invariant to projective transforms of $\mathbb{P}^3$ since the underlying object in the embedding is the Plucker line which itself is invariant to projective transforms of $\mathbb{P}^3$.

## 7 Conclusions & Future Work

In this paper we presented the use of the symmetric and anti-symmetric tensor algebras for exemplifying the geometry of linear and non-linear figures. We showed that the anti-symmetric algebra naturally encompasses the range of linear objects in $\mathbb{P}^2$ and $\mathbb{P}^3$, also providing us with a means producing projections between these spaces.

We broadened the application of the symmetric tensor algebra to include the representation of degree $d$ hypersurfaces, as far as the authors are aware this is first application of symmetric tensor algebra as a geometrically constructive operator in the computer vision literature. This understanding has allowed the generalization of earlier results for the projection of surfaces and also a novel operator for the projection of the Chow form of a curve in $\mathbb{P}^3$. These discoveries have resulted in a number of practical algorithms to compute location of curves and surfaces in space, as well as solving for the location of the cameras viewing known curves or surfaces, this will be a feature of future work.

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