Notes on Differential Forms

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1 Tensor Algebra

1.1 Manifolds and Local Coordinates

Let M be an n-dimensional smooth orientable manifold without boundary. Then locally, at any point $x_0 \in M$, there is a neighborhood such that it can be diffeomorphically mapped to a region in the Euclidean n-dimensional space $\mathbb{R}^n$ with the coordinates x^μ, where $\mu = 1, \dots, n$. What follows is a list of useful formulas in that local coordinate chart with these local coordinates.

1.2 Tangent and Cotangent Spaces

The tangent space $T_{x_0}M$ at the point x_0 is a vector space spanned by the basis

$$e_\mu = \partial_\mu = \partial/\partial x^\mu \tag{1}$$

(coordinate basis). A tangent vector v can be represented by a n-tuple v^μ, i.e.

$$v = v^\mu e_\mu . \tag{2}$$

The cotangent space $T^*_{x_0}M$ at the point x_0 is a vector space of linear maps

$$\alpha : T_{x_0}M \to \mathbb{R}, \qquad v \mapsto \langle \alpha, v \rangle , \tag{3}$$

spanned by the basis

$$\omega^\mu = dx^\mu \tag{4}$$

(coordinate basis). This basis is dual to the basis e_ν in the sense that

$$\langle \omega^\nu, e_\mu \rangle = \delta^\nu_\mu . \tag{5}$$

A cotangent vector α can be represented by a n-tuple α^μ; then

$$\alpha = \alpha_\mu \omega^\mu \tag{6}$$

and

$$\langle \alpha, v \rangle = \alpha_\mu v^\mu . \tag{7}$$

(Recall that a summation over repeated indices is performed.)

1.3 Tensors of Type (p, q)

A tensor of type (p, q) is a real valued multilinear map

$$A : \underbrace{T^*_{x_0}M \times \cdots \times T^*_{x_0}M}_{p} \times \underbrace{T_{x_0}M \times \cdots \times T_{x_0}M}_{q} \to \mathbb{R} \,. \tag{8}$$

A basis in the vector space of tensors of type (p, q) can be defined by

$$e_{\mu_1} \otimes \cdots \otimes e_{\mu_p} \otimes \omega^{\nu_1} \otimes \cdots \otimes \omega^{\nu_q} \,. \tag{9}$$

Then a tensor of the type (p, q) is represented by the components

$$A^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} \,, \tag{10}$$

so that

$$A = A^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} e_{\mu_1} \otimes \cdots \otimes e_{\mu_p} \otimes \omega^{\nu_1} \otimes \cdots \otimes \omega^{\nu_q} \,. \tag{11}$$

1.4 Riemannian Metric

A Riemannian metric is a symmetric tensor of the type $(0, 2)$ whose components $g_{\mu\nu}$ are given by a symmetric nondegenerate positive definite matrix $g_{\mu\nu}$. The Euclidean metric is given just by the Kronecker delta symbol, i.e.

$$g_{\mu\nu} = \delta_{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu, \\ 0 & \text{if } \mu \neq \nu \,. \end{cases} \tag{12}$$

The Riemannian metric defines an inner product of vectors by

$$(v, w) = g_{\mu\nu} v^\mu w^\nu \,, \tag{13}$$

and one-forms

$$(\alpha, \beta) = g^{\mu\nu} \alpha_\mu \beta_\nu \,, \tag{14}$$

where $g^{\mu\nu}$ is the matrix inverse to the matrix $g_{\mu\nu}$. It establishes an isomorphism between the tangent vectors and the covectors (one-forms) by

$$\alpha_\mu = g_{\mu\nu} v^\nu, \qquad v^\mu = g^{\mu\nu} \alpha_\nu \,. \tag{15}$$

Similarly, one defines the operation of raising and lowering indices of any tensor of type (p, q).

1.5 Differential Forms

A tensor α of type $(0, s)$ is called *skew-symmetric* or (*anti-symmetric*) if it changes sign when the order of any two of its arguments is reversed, i.e.

$$\alpha_{\dots\mu_i\dots\mu_j\dots} = -\alpha_{\dots\mu_j\dots\ \mu_i\dots}\,. \tag{16}$$

The skew-symmetric tensors of type $(0, p)$ (called *p-forms* or differential forms) form a subspace of

$$\underbrace{T^*_{x_0}M \otimes \cdots \otimes T^*_{x_0}M}_{p}\,. \tag{17}$$

For simplicity we will denote it by Λ_p.

Let S_p be the permutation group of integers $(1, \dots, p)$. The *signature* $\mathrm{sgn}(\sigma)$ (or *sign*) of a permutation $\sigma = \begin{pmatrix} 1 & \dots & p \\ \sigma(1) & \dots & \sigma(p) \end{pmatrix} \in S_p$ is defined to be $+1$ if σ is even and -1 if σ is odd. Then for any p-form α there holds

$$\alpha_{\mu_{\sigma(1)}\cdots\mu_{\sigma(p)}} = \mathrm{sgn}(\sigma)\alpha_{\mu_1\cdots\mu_p}\,. \tag{18}$$

Therefore, a p-form α is given by its components $\alpha_{\mu_1\cdots\mu_p}$ where

$$1 \le \mu_1 < \mu_2 < \cdots < \mu_{p-1} < \mu_p \le n\,. \tag{19}$$

The other components are given by symmetry, and symmetry gives no relations among the components with increasing indices. From this it is evident that the dimension of the space of p-forms in an n-dimensional manifold M is

$$\dim \Lambda_p = \binom{n}{p} \tag{20}$$

for any $0 \le p \le n$ and is zero for any $p > n$. In other words, $\Lambda_p = \{0\}$ if $p > n$. In particular, Λ_0 is one-diemsnional for $p = 0$ and $p = n$.

1.6 Exterior Product

For any tensor T of type $(0, p)$ we define the *alternating (or anti-symmetrization) operator* Alt . In components the antisymmetrization will be denoted by square brackets, i.e.

$$(\mathrm{Alt}\, T)_{\mu_1\cdots\mu_p} = T_{[\mu_1\cdots\mu_p]} = \frac{1}{p!}\sum_{\sigma\in S_p} \mathrm{sgn}(\sigma) T_{\mu_{\sigma(1)}\cdots\mu_{\sigma(p)}}\,, \tag{21}$$

where the summation is taken over the $p!$ permutations of $(1, \dots, p)$.

Since the tensor product of two skew-symmetric tensors is not a skew-symmetric tensor to define the algebra of antisymmetric tensors we need to define the *anti-symmetric tensor product* called the *exterior (or wedge) product*. If α is an p-form and β is an q-form then the wedge product of α and β is an $(p+q)$-form $\alpha \wedge \beta$ defined by

$$\alpha \wedge \beta = \frac{(p+q)!}{p!q!} \mathrm{Alt}\,(\alpha \otimes \beta)\,. \tag{22}$$

In components

$$(\alpha \wedge \beta)_{\mu_1 \dots \mu_{p+q}} = \frac{(p+q)!}{p!q!} \alpha_{[\mu_1 \dots \mu_p} \beta_{\mu_{p+1} \dots \mu_{p+q}]}\,. \tag{23}$$

The wedge product has the following properties

$$\begin{array}{ll} (\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) & \text{(associativity)} \\ \alpha \wedge \beta = (-1)^{\deg(\alpha)\deg(\beta)} \beta \wedge \alpha & \text{(anticommutativity)} \\ (\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma & \text{(distributivity)}\,, \end{array} \tag{24}$$

where $\deg(\alpha) = p$ denotes the degree of an p-form α.

A basis of the space Λ_p is

$$\omega^{\mu_1} \wedge \dots \wedge \omega^{\mu_p}\,, \qquad (1 \le \mu_1 < \dots < \mu_p \le n)\,. \tag{25}$$

An p-form α can be represented in one of the following ways

$$\begin{aligned} \alpha &= \alpha_{\mu_1 \dots \mu_p} \omega^{\mu_1} \otimes \dots \otimes \omega^{\mu_p} \\ &= \frac{1}{p!} \alpha_{\mu_1 \dots \mu_p} \omega^{\mu_1} \wedge \dots \wedge \omega^{\mu_p} \\ &= \sum_{\mu_1 < \dots < \mu_p} \alpha_{\mu_1 \dots \mu_p} \omega^{\mu_1} \wedge \dots \wedge \omega^{\mu_p}\,. \end{aligned} \tag{26}$$

The exterior product of a p-form α and a q-form β can be represented as

$$\alpha \wedge \beta = \frac{1}{p!q!} \alpha_{[\mu_1 \dots \mu_p} \beta_{\mu_{p+1} \dots \mu_{p+q}]} \omega^{\mu_1} \wedge \dots \wedge \omega^{\mu_{p+q}}\,. \tag{27}$$

1.7 Volume Element

The n-form
$$\varepsilon = \omega^1 \wedge \cdots \wedge \omega^n \tag{28}$$
is called the *volume element*. The components of the volume form denoted by
$$\varepsilon_{\mu_1 \dots \mu_n} \tag{29}$$
are given by so called *completely anti-symmetric Levi-Civita symbol* (or *alternating symbol*)

$$\varepsilon_{\mu_1 \dots \mu_n} = \begin{cases} +1 & \text{if } (\mu_1, \dots, \mu_n) \text{ is an even permutation of } (1, \dots, n), \\ -1 & \text{if } (\mu_1, \dots, \mu_n) \text{ is an odd permutation of } (1, \dots, n), \\ 0 & \text{otherwise}. \end{cases} \tag{30}$$

Furthermore, the space of n-forms Λ_n is one-dimensional. Therefore, any n-form α is represented as

$$\alpha = f\, \omega^1 \wedge \cdots \wedge \omega^n \,, \tag{31}$$

with some scalar f. The n-form

$$\sqrt{|g|}\, \omega^1 \wedge \cdots \wedge \omega^n \,, \tag{32}$$

where
$$|g| = \det g_{\mu\nu} \,, \tag{33}$$
and $g_{\mu\nu}$ is the Riemannian metric, is called the Riemannian volume element.

1.8 Interior Product

The *interior product* of a vector v and a p-form α is a $(p-1)$-form $i_v\alpha$ defined by
$$(i_v\alpha)_{\mu_1 \dots \mu_{p-1}} = v^\mu \alpha_{\mu\mu_1 \dots \mu_{p-1}} \,. \tag{34}$$

One can prove the following useful formula for the interior product of a vector v and the wedge product of a p-form α and a q-form β

$$i_v(\alpha \wedge \beta) = (i_v\alpha) \wedge \beta + (-1)^p \alpha \wedge (i_v\beta) \,. \tag{35}$$

1.9 The Star Operator (Duality)

The star operator $*$ maps any p-form α to a $(n-p)$-form $*\alpha$ defined by

$$(*\alpha)_{\mu_{p+1}\cdots\mu_n} = \frac{1}{p!}\varepsilon_{\mu_1\ldots\mu_p\mu_{p+1}\cdots\mu_n}\sqrt{|g|}g^{\mu_1\nu_1}\cdots g^{\mu_p\nu_p}\alpha_{\nu_1\ldots\nu_p}\,. \tag{36}$$

The operator $*$ satisfies an important identity: for any p-form α there holds

$$*^2\alpha = (-1)^{p(n-p)}\alpha\,. \tag{37}$$

Notice that if n is odd then $*^2 = 1$ for any p.

1.9.1 Examples ($\mathbb{R}^3$)

In the case of three-dimensional Euclidean space the metric is $g_{\mu\nu} = \delta_{\mu\nu}$, the bases of p-forms are:

$$1,\quad dx,\quad dy,\quad dz,\quad dx\wedge dy,\quad dx\wedge dz,\quad dy\wedge dz,\quad dx\wedge dy\wedge dz\,. \tag{38}$$

The star operator acts on this forms by

$$*1 = dx\wedge dy\wedge dz, \tag{39}$$

$$*dx = dy\wedge dz,\quad *dy = -dx\wedge dz,\quad *dz = dx\wedge dy, \tag{40}$$

$$*(dx\wedge dy) = dz,\quad *(dy\wedge dz) = dx,\quad *(dx\wedge dz) = -dy, \tag{41}$$

$$*(dx\wedge dy\wedge dz) = 1\,. \tag{42}$$

So, any 2-form

$$\alpha = \alpha_{12}dx\wedge dy + \alpha_{13}dx\wedge dz + \alpha_{23}dy\wedge dz \tag{43}$$

is represented by the dual 1-form

$$*\alpha = \alpha_{12}dz - \alpha_{13}dy + \alpha_{23}dx\,, \tag{44}$$

that is

$$(*\alpha)_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda}\alpha^{\nu\lambda} \tag{45}$$

$$(*\alpha)_1 = \alpha_{23}\,,\qquad (*\alpha)_2 = \alpha_{31}\,,\qquad (*\alpha)_3 = \alpha_{12}\,, \tag{46}$$

and any 3-form α

$$\alpha = \alpha_{123} dx \wedge dy \wedge dz \tag{47}$$

is represented by the dual 0-form

$$*\alpha = \frac{1}{3!}\varepsilon_{\mu\nu\lambda}\alpha^{\mu\nu\lambda} = \alpha_{123}\,. \tag{48}$$

Now, let α and β be two 1-forms

$$\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz\,, \qquad \beta = \beta_1 dx + \beta_2 dy + \beta_3 dz\,. \tag{49}$$

Then

$$*\beta = \beta_1 dy \wedge dz + \beta_2 dz \wedge dx + \beta_3 dx \wedge dz \tag{50}$$

and

$$\alpha \wedge \beta = (\alpha_1\beta_2 - \alpha_2\beta_1) dx \wedge dy + (\alpha_1\beta_3 - \alpha_3\beta_1) dx \wedge dz + (\alpha_2\beta_3 - \alpha_3\beta_2) dy \wedge dz\,, \tag{51}$$

$$\alpha \wedge (*\beta) = (\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3) dx \wedge dy \wedge dz\,. \tag{52}$$

Therefore,

$$*(\alpha \wedge \beta) = (\alpha_1\beta_2 - \alpha_2\beta_1) dz - (\alpha_1\beta_3 - \alpha_3\beta_1) dy + (\alpha_2\beta_3 - \alpha_3\beta_2) dx\,, \tag{53}$$

$$*[\alpha \wedge (*\beta)] = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3\,, \tag{54}$$

or

$$\boxed{*(\alpha \wedge \beta) = \alpha \times \beta}\,, \tag{55}$$

$$\boxed{*[\alpha \wedge (*\beta)] = \alpha \cdot \beta}\,. \tag{56}$$

2 Tensor Analysis

3 Exterior Derivative (Gradient)

The *exterior derivative* of a p-form is a $(p+1)$-form with the components

$$\begin{aligned} (d\alpha)_{\mu_1\dots\mu_{p+1}} &= (p+1)\,\partial_{[\mu_1}\alpha_{\mu_2\dots\mu_{p+1}]} \\ &= \sum_{q=1}^{p+1}(-1)^{q-1}\partial_{\mu_q}\alpha_{\mu_1\dots\mu_{q-1}\mu_{q+1}\dots\mu_{p+1}}\,. \end{aligned} \tag{57}$$

It is a linear map satisfying the conditions:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta \,, \tag{58}$$

$$d^2 = 0 \,. \tag{59}$$

For any n-form α (a p-form with rank equal to the dimension of the manifold $p = n$) the exterior derivative vanishes

$$d\alpha = 0 \,. \tag{60}$$

One can prove the following important property of the exterior derivative of the wedge product of a p-form α and a q-form β (product rule)

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge (d\beta) \,. \tag{61}$$

3.1 Examples in $\mathbb{R}^3$

Zero-Forms. For a 0-form f we have

$$(df)_\mu = \partial_\mu f \,, \tag{62}$$

so that

$$\boxed{df = \operatorname{grad} f} \,. \tag{63}$$

One-Forms. For a 1-form

$$\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz \tag{64}$$

we have

$$(d\alpha)_{\mu\nu} = \partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu \tag{65}$$

that is

$$d\alpha = (\partial_1\alpha_2 - \partial_2\alpha_1) dx \wedge dy + (\partial_2\alpha_3 - \partial_3\alpha_2) dy \wedge dz + (\partial_3\alpha_1 - \partial_1\alpha_3) dz \wedge dx \,. \tag{66}$$

Therefore

$$(*d\alpha)^\mu = \varepsilon^{\mu\nu\lambda} \partial_\nu \alpha_\lambda \,, \tag{67}$$

so that

$$*d\alpha = (\partial_2\alpha_3 - \partial_3\alpha_2) dx + (\partial_3\alpha_1 - \partial_1\alpha_3) dy + (\partial_1\alpha_2 - \partial_2\alpha_1) dz \,. \tag{68}$$

We see that

$$\boxed{*d\alpha = \mathbf{curl}\, \alpha} \,. \tag{69}$$

Two-Forms. For a 2-form β there holds

$$(d\beta)_{\mu\nu\lambda} = \partial_\mu \beta_{\nu\lambda} + \partial_\nu \beta_{\lambda\mu} + \partial_\lambda \beta_{\mu\nu}\,, \tag{70}$$

or

$$d\beta = (\partial_1\beta_{23} + \partial_2\beta_{31} + \partial_3\beta_{12}) dx \wedge dy \wedge dz\,. \tag{71}$$

Hence,

$$*d\beta = \frac{1}{2}\varepsilon^{\mu\nu\lambda}\partial_\mu\beta_{\nu\lambda} = \partial_1\beta_{23} + \partial_2\beta_{31} + \partial_3\beta_{12}\,. \tag{72}$$

Now let α be a 1-form

$$\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz\,. \tag{73}$$

Then

$$*\alpha = \alpha_1 dy \wedge dz - \alpha_2 dx \wedge dz + \alpha_3 dx \wedge dy\,, \tag{74}$$

and

$$d * \alpha = (\partial_1\alpha_1 + \partial_2\alpha_2 + \partial_3\alpha_3) dx \wedge dy \wedge dz\,, \tag{75}$$

or

$$*d * \alpha = \partial_1\alpha_1 + \partial_2\alpha_2 + \partial_3\alpha_3\,. \tag{76}$$

So,

$$\boxed{*d * \alpha = \operatorname{div}\alpha}\,. \tag{77}$$

3.2 Coderivative (Divergence)

Given a Riemannian metric $g_{\mu\nu}$ we also define the *co-derivative* of p-forms by

$$\delta = (-1)^{n(p-1)} * d * \,. \tag{78}$$

That is the coderivative of a p-form α is the $(p-1)$-form

$$\begin{aligned}(\delta\alpha)_{\mu_1\ldots\mu_{p-1}} &= \frac{(-1)^{n(p-1)}}{(n-p)!p!}\varepsilon_{\mu_p\ldots\mu_n\mu_1\ldots\mu_{p-1}}\sqrt{|g|}g^{\nu\mu_p}g^{\nu_{p+1}\mu_{p+1}}\cdots g^{\nu_n\mu_n} \\ &\quad\times\partial_\nu\left(\varepsilon_{\nu_1\ldots\nu_p\nu_{p+1}\ldots\nu_n}\sqrt{|g|}g^{\nu_1\lambda_1}\cdots g^{\nu_p\lambda_p}\alpha_{\lambda_1\ldots\lambda_p}\right)\end{aligned} \tag{79}$$

It is easy to see that, since $*^2 = \pm 1$ and $d^2 = 0$, the coderivative has the following property

$$\delta^2 = 0\,. \tag{80}$$

From this definition, we can also see that, for any 0-form f (a function) $*f$ is an n-form and, therefore, $d*f = 0$i.e. a coderivative of any 0-form is zero

$$\delta f = 0\,. \tag{81}$$

For a 1-form α, $\delta\alpha$ is a 0-form

$$\boxed{\delta\alpha = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\alpha_\nu\right)}\,. \tag{82}$$

More generally, one can prove that for a p-form α

$$(\delta\alpha)_{\mu_1\dots\mu_{p-1}} = g_{\mu_1\nu_1}\cdots g_{\mu_{p-1}\nu_{p-1}}\frac{1}{\sqrt{|g|}}\partial_\nu\left(\sqrt{|g|}g^{\nu\lambda}g^{\nu_1\lambda_1}\cdots g^{\nu_{p-1}\lambda_{p-1}}\alpha_{\lambda\lambda_1\dots\lambda_{p-1}}\right)\,. \tag{83}$$

4 Integration of Differential Forms

Any differential n-form α can be integrated over the n-dimensional manifold M. One needs to introduce an atlas of local charts with local coordinates that cover the whole manifold. For simplicity, we will describe the integrals over a single chart only. That is we have local coordinates x^μ that map a region in the manifold M to a bounded region U in the Euclidean space $\mathbb{R}^n$. This region is supposed to have some nice boundary ∂U. The the integral

$$\int_U \alpha = \int_U \alpha_{1\dots n}dx^1\wedge\cdots\wedge dx^n \tag{84}$$

is just an ordinary multiple integral over the coordinates $x^1,\dots,x^n$, in the usual notation

$$\int_U \alpha = \int_U \alpha_{1\dots n}(x)\,dx^1\cdots dx^n \tag{85}$$

More generally, any differential p-form α can be integrated over a p-dimensional submanifold N of an n-dimensional manifold M. Since N itself is a manifold this case reduces to the case of integration of a n-form over a n-diemsnional manifold. Clealy, it depends on the embedding of the submanifold N in the manifold M. If $x = (x^\mu) = (x^1,\dots,x^n)$, $\mu = 1,\dots,n$, are the local coordinates on the manifold M and $u = (u^1, \dots, u^m) = (u^j)$, $j = 1, \dots, p$, are the local coordinates of the submanifold N, then

$$\int_N \alpha = \int_N \alpha_{\mu_1 \dots \mu_p}(x(u)) \frac{\partial x^{[\mu_1}}{\partial u^1} \cdots \frac{\partial x^{\mu_p]}}{\partial u^p} \, du^1 \wedge \cdots \wedge du^p \,. \tag{86}$$

The general Stokes Theorem states that for any smooth $(n-1)$-form α defined over a bounded region U of a n-dimensional manifold M (in particular, of $\mathbb{R}^n$) with a piecewise simple (no self-intersection) smooth boundary ∂U the following formula holds

$$\int_U d\alpha = \int_{\partial U} \alpha \,. \tag{87}$$

Here it is assumed that the orientation of ∂U is consistent with the orientation of U. The same formula holds for orientable manifolds with boundary.

4.1 Examples

One-forms. If $\alpha = \alpha_\mu dx^\mu$ is a one-form and U is a curve $x^\mu = x^\mu(t)$, $a \le t \le b$, then

$$\int_U \alpha = \int_a^b \alpha_\mu(x(t)) \frac{dx^\mu(t)}{dt} dt \,. \tag{88}$$

Two-forms. If $\alpha = \frac{1}{2}\alpha_{\mu\nu} dx^\mu \wedge dx^\nu$ is a two-form and U is a surface $x^\mu = x^\mu(u)$, $u = (u^1, u^2) \in U$, then

$$\int_U \alpha = \int_U \frac{1}{2} \alpha_{\mu\nu}(x(u)) J^{\mu\nu}(x(u)) \, du^1 \wedge du^2 \,, \tag{89}$$

where

$$J^{\mu\nu} = e_1^\mu e_2^\nu - e_1^\nu e_2^\mu \,, \tag{90}$$

where e_1 and e_2 are tangent vectors to the surface defined by

$$e_j^\mu = \frac{\partial x^\mu}{\partial u^j} \,. \tag{91}$$

In three dimensional Euclidean space $\mathbb{R}^3$ one can represent the 2-forms α and J by their duals. The dual to the 2-form J is a one-form

$$*J = e_1 \times e_2 = n\sqrt{|g|}, \tag{92}$$

where n is the unit vector (normal to the surface since it is normal to both vectors e_1 and e_2), $|g| = \det g_{ij}$ and g_{ij} is the induced Riemannian metric on the surface defined as

$$\sum_1^3 (dx^\mu)^2 = g_{ij}(u) du^i du^j \,. \tag{93}$$

Therefore, the above formula simplifies to

$$\int_U \alpha = \int_U (*\alpha) \cdot n \sqrt{|g|} du^1 \wedge du^2 \,. \tag{94}$$