GEOMETRIC DIVISION OF NON-CONGRUENT QUANTITIES.

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Grassmann has treated the subject of division in the fourth chapter of his *Ausdehnungslehre* of 1844. He shows that the quotient of two of the quantities which he is treating is, in general, indefinite; but that, if the divisor and dividend are *congruent*, the quotient is definite, being simply the numerical ratio of their magnitudes. He has not, however, treated in general the ratio of two quantities of the same order, merely touching briefly, in the preface to the first edition, upon the ratio of two vectors. He says, in §141 of the same book: "*Der quotient stellt dann und nur dann einen einzigen, endlichen Werth dar, wenn der Divisor von geltendem Werthe ist, und zugleich entweder selbst als Grösse nullter Stufe dargestellt werden kann, oder dem Dividend gleichartig ist.*" By "*Werth*" he means an extensive quantity of some order, and with this meaning the statement is perfectly true; but I propose to show how a definite signification may be assigned to the quotient of two quantities of the same order, but not congruent, and also to consider the general quotient of two quantities, one of which is not a factor of the other.

We will take up first, however, one or two cases of quotients in which the divisor is a factor of the dividend, or *vice versa.*

The defining equation of division is

$$\frac{B}{A\cdot} A = B, \text{ or } A \frac{B}{\cdot A} = B;$$

using the dot as Grassmann does to indicate whether A is to be the first or second factor, i. e. whether A operates on $\frac{B}{A}$, or $\frac{B}{A}$ on A. We have, if p_1, p_2, etc. are points,

$$\frac{p_1 p_2 p_3}{p_1 p_2\cdot} = p_3 + x p_1 + y p_2;$$

for, multiplying both sides by $p_1 p_2$, we have the identity $p_1 p_2 p_3 = p_1 p_2 p_3$. As x and y may have any values whatever, the quotient is any point in the plane $p_1 p_2 p_3$; only, however, lying on the line $p_1 p_2$ when $x + y = \infty$. If $x + y = 0$, the quotient is $p_3 + x(p_1 - p_2)$, i.e. any point in a line through p_3 parallel to $p_1 p_2$. In plane space $p_1 p_2 p_3$ is a scalar quantity; therefore, dividing by it, we have the reciprocal of $p_1 p_2$, viz.:

$$\frac{1}{p_1 p_2} = \frac{1}{p_1 p_2 p_3}(p_3 + x p_1 + y p_2).$$

In this case, p_3 is any point not collinear with p_1 and p_2. Similarly,

$$\frac{1}{p_1} = \frac{1}{p_1 p_2 p_3}(p_2 p_3 + x p_3 p_1 + y p_1 p_2),$$

in which p_2 and p_3 are any two points not collinear with p_1. In solid space we have, similarly, $p_1 p_2 p_3 p_4$ being scalar, while $p_1 p_2 p_3$ is not so,

$$\frac{1}{p_1 \cdot} = \frac{1}{p_1 p_2 p_3 p_4}(p_2 p_3 p_4 + x p_3 p_4 p_1 + y p_4 p_1 p_2 + z p_1 p_2 p_3);$$

so that the reciprocal is any plane whatever, or more accurately, some *portion* of any plane whatever.

We will now consider quotients of quantities of the same order. Such a quotient will be itself of the *zero* order, i. e. in some sense a *numerical* quantity. If the divisor and dividend are congruent, the quotient is a mere number, viz. the ratio of their magnitudes. The ratios of points will first be discussed. We have $\frac{p_2}{\cdot p_1} p_1 = p_2 = \tau p_1$, say. Regarding $\frac{p_2}{\cdot p_1} (= \tau)$ as an operator, we see that its effect upon p_1 is to change it into p_2, i. e. to move p_1 a definite distance in a definite direction. τ is therefore a directed quantity, and yet numerical, as having no geometrical dimensions; we may call it a *directed number* in virtue of its character, and a *transferrer* in virtue of its office. Looking at the ratio of two points in this way, it appears that, although *directed*, it should yet, on account of its numerical nature, have no *positional* qualities; i. e., if $p_1 p_2$ and $p_3 p_4$ have the same direction and are equal in length, we may assume $\frac{p_2}{\cdot p_1} = \frac{p_4}{\cdot p_3}$, or $\tau_{12} = \tau_{34}$. Since we are to use these ratios as operators, we shall always write them to the left of the operand, and may omit the dot in the denominator. $\frac{p_2}{p_1} = \frac{p_4}{p_3}$ gives $p_2 = \frac{p_4}{p_3} p_1$, and $\frac{p_2}{p_1} p_3 = p_4$, but *not* $p_2 p_3 = p_1 p_4$; because $p_1 \frac{p_2}{p_1} p_3 = p_1 \left(\frac{p_2}{p_1} p_3\right)$, i. e. the quantity in parentheses is to be multiplied as a *whole* by p_1.

If τ is an operator which moves *any* point a definite distance in a given direction, then evidently $\tau(\tau p) = \tau^2 p$ is a point twice as far from p as τp, in the same direction. Similarly, τ^n moves p n times as far as τ, and it is easy to see that the exponential law holds good as for ordinary numbers, whether n be $+$, $-$, or fractional. Since $\tau^{-n}(\tau^n p) = p$, it appears that, if the exponent of τ be negative, the transference is in the opposite direction. We have $\tau^{-1}\tau = \tau\tau^{-1} = \tau^0 = \frac{\tau}{\tau} = 1$. Also, by Fig. 1,

$$\tau_1^{-1}\tau_2\tau_1 e = \tau_2\tau_1^{-1}\tau_1 e = \frac{\tau_2}{\tau_1}\tau_1 e = \tau_2 e;$$

but $$\frac{\tau_2 e}{\tau_1 e}\tau_1 e = \tau_2 e\,; \quad \therefore \frac{\tau_2 e}{\tau_1 e} = \frac{\tau_2}{\tau_1}.$$

And again, $\tau_2^{-1}\tau_1^{-1} = (\tau_2\tau_1)^{-1} = \frac{1}{\tau_1\tau_2}$. Thus the τ's are subject to the ordinary numerical laws of multiplication and division among themselves.

Also, if e be any point, $\tau e - e = (\tau - 1)e = \varepsilon$, say, is a vector; hence $\tau - 1$ is an operator that changes a point into a vector, i. e. moves it to infinity and reduces its weight to zero.

$(\tau - 1)^{-1}(\tau - 1)\,e = e = (\tau - 1)^{-1}\varepsilon$; hence $(\tau - 1)^{-1}$ is an operator which changes a vector into a point.

$\tau'(\tau - 1)\,e = \tau'\tau e - \tau' e = \tau e - e = (\tau - 1)\,e$; hence the τ operator has no effect on a vector.

Of course $\tau_m - \tau_n$ is an operator of the same kind as $\tau - 1$.

By the figure,
$$\tau_1 + \tau_2 = 2\tau_1^{\frac{1}{2}}\tau_2^{\frac{1}{2}},$$
$$\therefore \tau_1^2 + 2\tau_1\tau_2 + \tau_2^2 = 4\tau_1\tau_2,$$
$$\therefore (\tau_1 - \tau_2)^2 = 0;$$

i. e. the operator $\tau_2 - \tau_1$ reduces a vector to zero.

We have, also, $$\tau^n e = e + n(\tau e - e) = (1 + n\tau - n)\,e;$$
$$\therefore \tau^n = 1 - n + n\tau;$$

from which the numerical character of τ is apparent.

Since the sum of n unit points is n times the mean point of the system, we have

$$\tau_1 e + \tau_2 e + \ldots + \tau_n e = \sum_1^n \tau \,.\, e = n\bar{\tau} e; \quad \therefore \frac{1}{n}\sum_1^n \tau = \bar{\tau} \tag{1}$$

is an operator which moves a point to the mean of the positions to which τ_1, τ_2, etc. move it.

Now if $\tau_1, \tau_2, \ldots \tau_n$ be successively applied to the same point, i. e. the point be multiplied by τ_1, this product by τ_2, etc., the final position of the point will be in the right line through the original point and the mean of all the points $\tau_1 e, \tau_2 e$, etc., and n times as far from the original point as the mean point is; hence

$$(\tau_n\tau_{n-1}\ldots\tau_1)^{\frac{1}{n}} = (\tau_1\tau_2\ldots\tau_n)^{\frac{1}{n}} = \bar{\tau} = \frac{1}{n}\sum_1^n \tau\,; \tag{2}$$

i. e. the arithmetical and geometrical means of a series of τ-operators are equal.

It may be easily shown that we have also more generally

$$\frac{\sum_1^n (m\tau)}{\sum_1^n m} = \left(\tau_1^{m_1}\tau_2^{m_2}\ldots\tau_n^{m_n}\right)^{\frac{1}{\Sigma m}}. \tag{3}$$

Let ϵ be the base of Naperian logarithms, and write

$$\epsilon^{\tau} = 1 + \tau + \frac{1}{2\,!}\tau^2 + \frac{1}{3\,!}\tau^3 + \ldots, \tag{4}$$

in which the right hand member is to be the interpretation of the left. Then, by (3),

$$\epsilon^{\tau} = \left(2 + \frac{1}{2\,!} + \frac{1}{3\,!} + \ldots\right)\left((\tau^2\tau^{\frac{1}{2}}\tau^{\frac{1}{3\,!}}\tau^{\frac{1}{4\,!}}\ldots\right)^{\frac{1}{2+\frac{1}{2}+\frac{1}{3\,!}+\ldots}} = \epsilon\tau. \tag{5}$$

$$\therefore \log \epsilon\tau = 1 + \log \tau = \tau, \text{ or } \log \tau = \tau - 1. \tag{6}$$

Therefore $\log \tau$ is an operator that changes a point into a vector. From (5)

$$\tau = \epsilon^{-1}\epsilon^{\tau} = \epsilon^{\tau-1}. \tag{7}$$

Let us next consider the equation

$$\tau = \tau_1^{\,x}\tau_2^{\,y}\tau_3^{\,z}, \tag{8}$$

in which τ_1, τ_2, τ_3 are non-coplanar unit transferrers, x, y, z scalar variables, and τ a variable transferrer. If τ of equation (8) operate on any point e, it will transfer e to any point of space, when suitable values are given to x, y, z.

If one relation connect x, y, z, then τe will be a point on some surface; if two relations subsist between the scalar variables, the locus of τe is a curve. Suppose x, y, z to be all functions of t; by equation (6) equation (8) becomes

$$\log \tau = \tau - 1 = x \log \tau_1 + y \log \tau_2 + z \log \tau_3;$$

$$\therefore \frac{d\tau}{dt} = \frac{dx}{dt}\log \tau_1 + \frac{dy}{dt}\log \tau_2 + \frac{dz}{dt}\log \tau_3 = \log \tau_1^{\frac{dx}{dt}}\tau_2^{\frac{dy}{dt}}\tau_3^{\frac{dz}{dt}}. \tag{9}$$

By (6) $\frac{d\tau}{dt}$ changes e into a vector, and this vector is evidently parallel to the tangent at the point τe; hence for the equation of the tangent line, we have

$$\tau' = \tau + \frac{d\tau}{dt}w. \tag{10}$$

If x and y are independent, we have

$$\frac{d\tau}{dx} = \log \tau_1\tau_3^{\frac{dz}{dx}} \quad \text{and} \quad \frac{d\tau}{dy} = \log \tau_2\tau_3^{\frac{dz}{dy}}; \tag{11}$$

and for the tangent plane to a surface,

$$\tau' = \tau + \frac{d\tau}{dx}u + \frac{d\tau}{dy}v. \tag{12}$$

For example, the equation of the hyperbola may be written, omitting the e from both sides,

$$\tau = \tau_1^t \tau_2^{t-1}; \tag{13}$$

whence for the tangent

$$\begin{aligned} \tau' &= \tau_1^t \tau_2 \tau^{t-1} + \log \tau_1^w \tau_2^{-wt-2} \\ &= \tau_1^t \tau_2^{t-1} + \tau_1^w \tau_2^{-wt-2} - 1, \text{ by (6)}, \\ &= \tau_1^{t+w} \tau_2^{(t-w)t-2}, \text{ by (3)}. \end{aligned} \tag{14}$$

Similarly for the helix, we have

$$\tau = \tau_1^{a\cos\theta}\, \tau_2^{a\sin\theta}\, \tau_3^{c\theta}; \tag{15}$$

and for the tangent plane,

$$\tau' = \tau_1^{a\,(\cos\theta - w\sin\theta)}\, \tau_2^{a\,(\sin\theta + w\cos\theta)}\, \tau_3^{c\,(\theta + w)}. \tag{16}$$

From the equation $\frac{\tau_2 - 1}{\tau_1 - 1}(\tau_1 - 1)\, e = (\tau_2 - 1)\, e$, it appears that $\frac{\tau_2 - 1}{\tau_1 - 1}$ is an operator which changes the vector $(\tau_1 - 1)\, e$ into the vector $(\tau_2 - 1)\, e$; i. e. turns the first vector through a certain angle and changes its length in the ratio of the lengths of τ_1 and τ_2. When the lengths of τ_1 and τ_2 are the same, we will write

$$\frac{\tau_2 - 1}{\tau_1 - 1} = \mathfrak{v} = \frac{\varepsilon_2}{\varepsilon_1}, \tag{17}$$

if we make $\varepsilon_2 = (\tau_2 - 1)\, e$ and $\varepsilon_1 = (\tau_1 - 1)\, e$.

The expression in (17) is a *versor* for which reason we use the German $\mathfrak{v}$ to represent it.

$\frac{\varepsilon}{e}$ is an operator which changes the point e into the vector ε, and if $\varepsilon = (\tau - 1)\, e$, we may write

$$\frac{\varepsilon}{e} = \tau - 1. \tag{18}$$

Similarly,

$$\frac{e}{\varepsilon} = (\tau - 1)^{-1}. \tag{19}$$

The ratio of two points has been treated by Unverzagt in his *Theorie der goniometrischen und longimetrischen Quaternionen*, but in a manner quite different from the above.

We will now consider the ratio of two vectors purely as a *versor*, not giving to such ratios the compound character of versor and vector as Hamilton did, whereby his calculus became necessarily one of complexes.

In the first place, in *plane* space, the versor is *scalar* in character, since rotation can occur only in one plane, and therefore two rotations can differ only in magnitude and sign. In fact, $\mathfrak{v}$ is in this case simply $(\sqrt{-1})^x$.

In solid space, however, if $\mathfrak{v} = \frac{\varepsilon_2}{\varepsilon_1}$, and $T\varepsilon_2 = T\varepsilon_1$, $\mathfrak{v}$ turns any vector parallel to the plane $\varepsilon_1\varepsilon_2$ through an angle equal to that from ε_1 to ε_2 and in the same direction; i. e. the rotation is about an axis perpendicular to the plane $\varepsilon_1\varepsilon_2$. Hence, in this case, $\mathfrak{v}$ like τ, is a directed number. It is still essentially $(\sqrt{-1})^x$, with a quality of direction added.

Let us consider the general effect of the operator $\mathfrak{v}$ on any vector ε. Since $\mathfrak{v}$ is purely a versor, it ought not to change in any way the *character* of ε; that is, $\mathfrak{v}\varepsilon$ remains a *vector*. Suppose ε is parallel to the axis of $\mathfrak{v}$; then, as neither ε nor $\mathfrak{v}$ has particular *position*, they simply having a common point at infinity, and as $\mathfrak{v}$ can produce no change of *direction* of ε about the axis of $\mathfrak{v}$ to which ε is parallel, it appears that $\mathfrak{v}$ should have *no effect whatever* upon ε; that is, when ε is parallel to the axis of $\mathfrak{v}$, $\mathfrak{v}\varepsilon = \varepsilon$.

Next suppose ε and $\mathfrak{v}$ (Fig. 2) to make some angle between 0 and 90° with each other. Take ε' perpendicular to $\mathfrak{v}$ and ε'' parallel to $\mathfrak{v}$, so that $\varepsilon = \varepsilon' + \varepsilon''$;

then $$\mathfrak{v}\varepsilon = \mathfrak{v}(\varepsilon' + \varepsilon'') = \mathfrak{v}\varepsilon' + \mathfrak{v}\varepsilon'',$$

assuming the distributive law. But we have just seen that

$$\mathfrak{v}\varepsilon'' = \varepsilon''; \quad \therefore \mathfrak{v}\varepsilon = \varepsilon'' + \mathfrak{v}\varepsilon',$$

and $\mathfrak{v}$ revolves ε *conically* through some angle θ about the axis of $\mathfrak{v}$. Thus $\mathfrak{v}$ exactly corresponds to Hamilton's $q(\ )q^{-1}$.

We see thus to what we are led, if we preserve the idea of *geometric dimensions* by making the ratio of two vectors essentially numerical, and an operator distinct from a vector which is a geometric magnitude of the first order.

The question arises, how it is that Hamilton was led to make the assumption that the properties of versor and vector *must* be combined in the same quantity, as asserted in Article 64 of Tait's Quaternions. The gist of the whole matter is in this, that Hamilton assumed that *any given operator must always have the same meaning, or effect, independently of the operand;* something that is by no means axiomatic.

Let $\iota_1, \iota_2, \iota_3$ (Fig. 3) be three mutually rectangular unit vectors, and $\dot{i}_1, \dot{i}_2, \dot{i}_3$ three correspondingly parallel versors, as in the figure, so that

$$\frac{\iota_3}{\iota_2} = \dot{i}_1, \quad \frac{\iota_1}{\iota_3} = \dot{i}_2, \quad \frac{\iota_2}{\iota_1} = \dot{i}_3.$$

Then because $-\iota_2 = \dot{i}_1\iota_3 = \dot{i}_1(\dot{i}_1\iota_2) = \dot{i}_1^2\iota_2$, Hamilton assumes that the square of a versor must always and everywhere be -1, without reference to what is operated upon. But, according to the natural interpretation we have already arrived at from considering versors purely as such, for the case of a versor operating on a parallel vector, we should have $i_1^2\iota_1 = i_1(i_1\iota_1) = i_1\iota_1 = \iota_1$; that is, with ι_1 for operand, $i_1^2 = 1$ instead of -1.

Again, consider the operator $i_1 i_2 i_3$; we have

$$i_1 i_2 i_3 \iota_2 = i_1 i_2(-\iota_1) = i_1\iota_3 = -\iota_2,$$

so that, in this case, the versor is equivalent to -1, and its value is assumed by Hamilton to be therefore necessarily always -1. Take, however, ι_1 and ι_2 successively as operands, and we find

$$i_1 i_2 i_3 \iota_1 = i_1 i_2 \iota_2 = i_1 \iota_2 = \iota_3 = i_2^{-1}\iota_1; \therefore \text{ in this case } i_1 i_2 i_3 = i_2^{-1};$$
$$i_1 i_2 i_3 \iota_3 = i_1 i_2 \iota_3 = i_1 \iota_1 = \iota_1 = i_2 \iota_3; \quad \therefore \text{ in this case } i_1 i_2 i_3 = i_2.$$

In order to make an algebra in which $i_1^2 = i_1 i_2 i_3 = -1$ invariably, it was unavoidably necessary to assume $i_1 = \iota_1$ etc., thus introducing complexity, and ignoring completely the useful and convenient idea of geometric dimensions. The fact that Grassmann's whole system is founded upon this idea of dimensions, is that which renders it so much more natural, simple, and practical than quaternions.

Quotient of two lines (point-vectors) in plane space. We have, if L_1 and L_2 are point-vectors; i. e. each is the product of two points, or of a point and a vector, $\frac{L_2}{L_1} L_1 = L_2$; so that $\frac{L_2}{L_1}$ is an operator which changes L_1 into L_2. Call such an operator Λ, so that $\Lambda L = L'$; the question arises how to determine L'; i. e. what is the effect of Λ on other lines besides the one that appears in the denominator of its value. A natural interpretation of the meaning of ΛL may be arrived at as follows:—

Using the sign $|$ as Grassmann does to signify *complement*, write $L = |\,p$, and then assume $\Lambda L = |\,\tau p$. This will enable us to construct ΛL with ease. We have at once $(\Lambda - 1) L = |\,(\tau - 1)p$; but $(\tau - 1)p$ is a point at ∞ in a definite direction' and hence $(\Lambda - 1) L$, its complement, must be a line through the mean point of the reference triangle. Thus, as to every τ there corresponds a certain point at infinity in a definite direction from p, so to every Λ there corresponds a certain line through the mean of the reference points, cutting L in a definite point.

Since we have $(\tau^n - 1)p = n(\tau - 1)p$, we have also the complementary equation

$$(\Lambda^n - 1) L = n(\Lambda - 1) L, \tag{20}$$

by the aid of which, if we know ΛL, we can at once construct $\Lambda^n L$, as in Fig. 4. It is evident that all the relations between τ operators shown in equations (1) to (7) hold equally for Λ operators.

Fig. 5 is a diagram corresponding to Fig. 1 for points. e_0 is the mean point of the reference system, the three points of this system not being shown, as they are unnecessary; it corresponds to the line at infinity in the τ system. To A_1 there belongs the line $(A_1 - 1)L$, coinciding with e_0e_1; to A_2 there belongs, in the same way e_0e_2; and to A_1A_2 belongs e_0e_{12}. These correspond to the points at infinity in the directions of τ_1, τ_2, and $\tau_1\tau_2$ of Fig. 1.

As in Fig. 1, $e, \tau_1\tau_2e, (\tau_1\tau_2)^{\frac{1}{2}}e$ are collinear, so in Fig. 5, L, A_1A_2L, and $(A_1A_2)^{-} L$ pass through one point e_{12}. As in Fig. 1, $(\tau_1\tau_2)^{\frac{1}{2}}e = \frac{1}{2}(\tau_1 + \tau_2)e$, so in Fig. 5, $(A_1A_2)^{-\frac{1}{2}}L = e_{12}d = \frac{1}{2}ef = \frac{1}{2}(A_1^{-1} + A_2^{-1})L$. To make the construction clearer, the following relations are given: $L = e_1a = e_2b = e_{12}c$; $A_1A_2L = lk = ih = e_{12}g$; $A_1L = lm$; $A_2L = in$.

Writing now the equation

$$A = A_1^{x}A_2^{y}, \tag{21}$$

A is an operator which by giving suitable values to x and y will move L to any position whatever in the plane space under consideration. If some relation subsist between x and y then A causes L to be always tangent to some curve. If A, A_1, A_2, L correspond reciprocally to τ, τ_1, τ_2, e, then when $y = f(x)$, (21) may be called the line equation of the curve reciprocal to that represented by the point equation $\tau = \tau_1^{x}\tau_2^{y}$. Furthermore, $\frac{dA}{dx} = \log A_1A_2^{\frac{dy}{dx}}$ is an operator which makes L pass through the point of contact of AL, and also through the mean of the reference points. Hence

$$A' = A + z\frac{dA}{dx} \tag{22}$$

is the equation of the point of contact of the line AL.

Thus
$$A = A_1^{t}A_2^{t^{-1}} \tag{23}$$

is the equation of the curve reciprocal to the hyperbola of equation (13), and of course represents different conics according to the position of the mean point with reference to that hyperbola. The equation

$$A' = A_1^{t+z}A_2^{(t-z)t^{-2}} \tag{24}$$

represents the point of contact. Since the line $A^{\infty}L$ coincides with the line $(A - 1)L$, and since in (23) the exponent of A_1 is ∞ when that of A_2 is 0, and *vice versa*, it appears that $(A_1 - 1)L$ and $(A_2 - 1)L$ are tangent to the curve enveloped by (23) at the points LA_1L and LA_2L. This corresponds to the statement that the points at infinity, $(\tau_1 - 1)e$ and $(\tau_2 - 1)e$, are the points of contact of the tangents, $e\tau_1e$ and $e\tau_2e$, to the curve represented by (13), i. e. of the asymptotes.

Quotient of two plane-vectors and of two point-plane-vectors in solid space. Let η_1 and η_2 be two plane vectors, i. e. each is the product of two line vectors. Then $\frac{\eta_2}{\eta_1}$ changes the plane-vector η_1 into η_2; i. e. revolves η_1 through a certain angle about some axis parallel to η_1 and η_2. Thus the operation is identical with that performed by the ratio of two vectors. Hence we may use the same letter $\mathfrak{v}$ to designate this operator. If $\mathfrak{v}$ operate on a plane vector η not parallel to its axis, then η may be resolved into two components, one parallel and one perpendicular to the axis of $\mathfrak{v}$; the parallel component will be rotated through the angle of $\mathfrak{v}$, while no effect can be produced on the other component; hence the total effect of $\mathfrak{v}$ on η will be to rotate it conically about the axis of $\mathfrak{v}$ through a definite angle.

Next let P_1 and P_2 be two planes or point-plane-vectors; i. e. each is the product of three points. Then

$$\frac{P_2}{P_1} = \Pi_{12}\text{, say,} \tag{25}$$

is an operator that changes P_1 into P_2. Now suppose the Π operators to play the same part in solid space that the Λ operators do in plane space; i. e. if $P = | p$, then $\Pi P = | \tau p$. Thus we have a reciprocal system in three dimensional space.

The equation
$$\Pi = \Pi_1^x \Pi_2^y \Pi_3^z \tag{26}$$

may represent any plane whatever; i. e. Π will move a given plane P to any position by giving suitable values to x, y, z. If one relation subsist between x, y, z, (26) will represent some convex or some skew surface, or else some curve. If two relations subsist between them, then (26) will represent a developable surface. If one of the scalars x, y, z be constant, and the others be connected by some equation, then (26) will represent a cone.

Of course, on the assumption above as to the nature of the Π operators, they will satisfy the same equations of relation as have been proved for the τ operators. To every Π corresponds a plane $(\Pi - 1) P$ through the mean point of the reference system and cutting P in a fixed line, and the construction of any particular value of ΠP in (26) is easy theoretically, though the diagram would be pretty complex.

We may now, by the aid of these operators, express the quotient of any two geometric quantities. Take, for instance, $\frac{p_1 p_2 p_3}{p_4}$. If the points are all coplanar, we have $p_1 p_2 p_3 = n p_1 p_2 p_4$, where n is some scalar multiplier.

Thus
$$\frac{p_1 p_2 p_3}{p_4} = \frac{n p_1 p_2 p_4}{p_4} = n p_1 p_2 + x p_2 p_4 + y p_4 p_1.$$

If, however, the points are not coplanar, we may write

$$\frac{p_1 p_2 p_3}{p_4} = p_1 p_2 \tau_{43} + x p_2 p_4 + y p_4 p_1 + z p_3 p_4;$$

τ_{43} being the operator that transfers p_4 to p_3. Again, take $\frac{p_3 p_4 p_5}{p_1 p_2}$. If the points are coplanar, we have

$$p_4 = m_1 p_1 + m_2 p_2 + m_3 p_3,$$
$$p_5 = n_1 p_1 + n_2 p_2 + n_3 p_3;$$
$$\therefore\ p_4 p_5 = \begin{vmatrix} p_2 p_3 & p_3 p_1 & p_1 p_2 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix},$$

and

$$p_3 p_4 p_5 = \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} p_1 p_2 p_3.$$

Hence the quotient becomes

$$\begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} \frac{p_1 p_2 p_3}{p_1 p_2} = \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} (p_3 + x p_1 + y p_2).$$

If the points are non-coplanar, we may write

$$p_5 = n_1 p_1 + n_2 p_2 + n_3 p_3 + n_4 p_4;$$

whence

$$p_3 p_4 p_5 = n_1 p_3 p_4 p_1 + n_2 p_3 p_4 p_2;$$

and the required quotient is

$$\frac{n_1 p_1 p_3 p_4 + n_2 p_2 p_3 p_4}{\cdot p_1 p_2} = p_4 (n_1 A_{12\,.\,13} + n_2 A_{12\,.\,23}) + x p_1 + y p_2.$$

In a similar way other cases may be treated.