Whitehead's Universal Algebra

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1. Introduction

Mathematics has long been regarded as a very special kind of language. Centuries ago, Galileo Galilei argued that the great book of Nature – which lies open before us in the form of the universe - is written in the mathematical language, such that "we cannot understand it if we do not first learn the language and grasp the symbols in which it is written". Its symbols, he stated, are those of geometry, "without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth." [1] Galileo's words express a faith that Euclid's ideal geometrical truths mirror the spatial geometry of the physical Universe. Newton's discoveries marked the triumph of this Galilean faith. However, in the nineteenth century a number of developments began to undermine this scientific edifice, calling into question the faith that Euclidian geometry was peculiarly fitted to reveal the truths of Nature.

When the young Alfred Whitehead arrived at Cambridge, in 1880, the scientific model of the world associated with the names of Galileo and Newton was still largely unquestioned. At that time, according to Whitehead, nearly all mathematicians shared the view of Plato and Euclid that "there was only one coherent analysis of the notion of space", and assumed that the aim of mathematics "was the adequate expression of this unique, coherent notion of spatiality". That view, Whitehead recalled, was overturned by a few mathematicians, who had invented "fantastic variations from orthodox geometry", and the subsequent discovery, "between 1890 and 1910, . . . that these variant types of geometry were of essential importance for the expression of our modern scientific knowledge." [2] One of the mathematicians that contributed to this revolution was Whitehead.

In 1890 Whitehead began to write A Treatise on Universal Algebra With Applications (1898) [UA]. This is the work in which the mathematical architecture of his post-Euclidian, post-Newtonian scientific vision is first articulated. Although best known in the mathematical community for his co-authorship, with Bertrand Russell, of the three volumes of Principia Mathematica (1910,1912,1913) [PM], and in philosophical circles for his metaphysical system, elaborated in Process and Reality (1927) [PR], it was UA that first brought Whitehead national and international recognition.[3] Consisting of over five hundred pages of mathematical notation, the text is far from inviting, and a cursory glance at UA conveys little of the intellectual drama implicit in its unfolding chapters. Hank Keeton, however, has argued that the "importance of UA cannot be overstated" in that it is Whitehead's first expression of the vision that "set the tone for his intellectual life's work." [4]

My goal in this chapter is to outline the historical context of the project that Whitehead develops in UA, review its contents, and survey its influence on the mathematical community. The central theme is Whitehead's quest for a new geometrical language for physics, and the mathematical vision that underpinned this ambition. In the concluding section I offer a few comments on the metaphysical orientation of UA.

2. The Context of UA: Multiple Algebra and Maxwell's New Physics

Whitehead's interest in Mathematical Logic was initially motivated and guided by his quest for a mathematical language suitable for expressing Maxwell's vision of matter and space. As he acknowledged in 1912, his "logical scrutiny of mathematical symbolism and mathematical ideas", beginning in 1890, "had its origin in the study of the mathematical theory of Electromagnetism, and has always had as its ultimate aim the general scrutiny of the relations of matter and space". [5] This practical motivation is essential to an appreciation of Whitehead's interest in Universal Algebra.

Whitehead had entered Cambridge at a crucial time in the development of modern physics. In the early eighties, Cambridge was a center for those interested in Maxwell's groundbreaking Treatise on Electricity and Magnetism. In 1884 Whitehead completed a dissertation on Maxwell's Treatise, and as a result gained a teaching fellowship at Cambridge's Trinity College. No copy of Whitehead's dissertation has survived, however in later years he recalled that Maxwell's crucial contribution was the elimination of the notion of empty space; all space was filled with ether. Maxwell's theory had extended and provided a mathematical formulation of Michael Faraday's theory of electrical and magnetic lines of force. Fundamental to Maxwell's interpretation of these lines of force was his rejection of attempts to reaffirm the traditional notion of forces acting at a distance, without mediation of a 'field'. As P. M. Harmann has recently reaffirmed, it was the notion of the ether, serving as the "seat of the electromagnetic field, that was the keystone of his theory." [6] It was this notion, Whitehead relates, that provided the basis for a new vision, in which matter represented entanglements in the ether – knots – that impose "stresses and strains throughout the whole jelly-like ether", such that agitations of ordinary matter are transmitted (as light, heat, electrical or magnetic energy) "through the ether as agitation of the stresses and strains." [7]

In Whitehead's early years at Cambridge, the dominant notion of the ether (due more to Thomson than Maxwell) was of a near perfect liquid, which for practical purposes could be regarded as frictionless, continuous and infinitely divisible. The ultimate 'atoms' of this liquid, it was hypothesized, were analogous to vortex-rings, and the coarser grained matter of our physical universe were constellations of these vortex-rings knotted together to form dynamic enduring patterns of activity. The slightest movement in such a corpuscle will send a shudder through the mediating liquid to the extremities of the universe – this electromagnetic shudder is perceived as light. Thus, for practical purposes, the vortexes in the ether could be viewed as an infinite manifold of contiguous points – perfect for the geometrical formulation of models of the universe – but leaving open the question of what kind of geometrical space was best suited to modeling this domain.

During Whitehead's first decade at Cambridge a number of prominent mathematicians and physicists were devoting attention to simplifying Maxwell's mathematical formulation of his theory, experimentally verifying key propositions, and exploring the philosophical implications of these developments.[8] Whitehead followed these developments avidly and incorporated his findings into his courses. In his early years at Trinity he taught most branches of applied mathematics, but developed a special interest in new forms of geometry, and their scientific applications. One of his students recalls that in the early 1890s he had "a place apart among our teachers, chiefly because his philosophic urge to grasp the nature of mathematics in its widest aspects led him to study what were at that time considered out-of-the-way branches of the subject" such as "Non-Euclidian Geometry".[9] This was a subject that had come to assume a practical significance for interpreters of Maxwell.

Maxwell expressed his theory of dynamics in terms of projective geometry (he referred to it as the "geometry of position").[10] Felix Klein, by building on the work of Arthur Cayley, had shown that the varied systems of Euclidian and Non-Euclidian Geometries could be viewed as more restricted variants of the basic set of transformations that define Projective Geometry.[11] Klein's system reinforced the view that Projective Geometry was the real foundation of spatial analysis, and provided a projective alternative to the metrical interpretation of non-Euclidian geometry.[12] Klein had visited England in the seventies, and during the eighties was a frequent guest at Trinity, enjoying the hospitality of Whitehead's good friend and mathematical colleague, Andrew Forsyth.[13] Whitehead was deeply influenced by Klein's standpoint, and became part of a circle of British mathematicians exploring non-Euclidian geometry from a projective viewpoint. The member of this circle who exercised the greatest influence upon the development of this field of research was Cayley's former student, William Clifford.[14]

During the 1870s, Clifford began to study the 1844 edition of Grassmann's Ausdehnungslehre and for the first time it lay in the hands of a British mathematician who grasped the significance of its ideas.[15] Clifford was moved to express his "profound admiration for that extraordinary work, and my conviction that its principles will exercise a vast influence upon the future of mathematical science." [16] In Grassmann, Clifford found the key to classifying and generalizing the results of Geometric Algebras. Behind his analysis is a concern to demonstrate how Hamilton's quaternions and his own 'bi-quaternions' can be expressed in terms of simpler laws and to show how these laws can clarify the form of the "two kinds of vectors which he [Maxwell] calls force and flow". Clifford had planned to publish a series of papers on the application of Grassmann's method, but tragically, died in the following year, at the age of 34.[17] Despite his death, his vision exerted a powerful influence upon a number of mathematicians through the eighties, including Whitehead.

Although Clifford's place in the history of mathematics has tended to be neglected, his insights concerning the relationship between Hamilton's quaternions and Grassmann's theory of extension are, in the words of Helmuth Malonek, “the key to a fair appreciation of Hamilton’s work as well as a milestone in the progress of this field of algebra.” [18] Clifford was described by one of his contemporaries as the young “lion” of the new generation of British mathematicians.[19] He had developed a deep interest in Maxwell’s work, and played a lively role challenging Euclidian models of space and suggesting alternative geometrical models of the cosmos (prefiguring Einstein’s geometrical explanations of space and matter). He argued that the work of Lobatchewsky and his successors, by dethroning Euclidian geometry and its notion of eternal truth, had liberated the mind from the fallacy of theoretical exactness, strengthening our respect for the evidence of experience, and appreciation of the practical, conditional nature of all truth.[20] Clifford extended Hamilton’s Quaternion algebra from three to four dimensions, thereby inventing Bi-Quaternion algebra in which the fundamental entities were represented by four rather than three vectors perpendicular to each other. This discovery revealed a simple and systematic way that his scheme could be extended to any number of dimensions, opening the way to a new appreciation of Grassmann’s algebra, and the mathematical language now referred to as Geometric Algebra.[21]

Clifford’s accessible style of writing made his introductory texts on mathematics and dynamics the logical point of entry for students in that era.[22] His enthusiastic endorsement of Grassmann marked a turning point in Grassmannian studies in Britain. Several other young members of Klein’s British circle, including a contemporary of Whitehead’s at Trinity, Homersham Cox, and Arthur Buchheim, an Oxford graduate, were soon to publish papers that became a vital source of inspiration for Whitehead. In 1882 Cox published a long paper, “On the Application of Quaternions and Grassmann’s Ausdehnungslehre to Different Kinds of Space”, the object of which was “following Grassmann, to establish a pure algebraical calculus, the laws of which will coincide with those of actual geometry.” [23] Buchheim, after spending time studying under Klein in Leipzig, published in 1884 the first of a series of papers, “The Theory of Screws in Elliptic Space”, which was intended to show that the “Ausdehnungslehre furnishes a complete explanation of the theory of screw and the theory of linear complex”, demonstrating that “they are identical and not merely analogous”.[24] Whitehead’s first acquaintance with Grassmann probably came from these British interpreters of Grassmann.[25] By the mid 80s he had turned to Grassmann’s original text, and in 1887 announced a course of lectures, open to students from all colleges, on “Grassmann’s Ausdehnungslehre, with special references to its applications”.[26] This was the first lecture course devoted to Grassmann’s work at Cambridge. After the course came the book.

3. The Content of UA

In 1888, one of the founders of modern vector theory, J. Willard Gibbs, had suggested that a struggle for existence was “commencing between the different methods and notions of multiple algebra, especially between the ideas of Grassmann and Hamilton.” [27] Whitehead’s work was intended to settle this dispute. But this was a bi-product of a larger design – that of laying bare the architecture of the new field of algebras that Hamilton and Grassmann had helped pioneer. Hence, the stated aim of UA was to “present a thorough investigation of the various systems of Symbolic Reasoning allied to ordinary Algebra”, as exemplified in such systems as “Hamilton’s Quaternions, Grassmann’s

Calculus of Extension, and Boole's Symbolic Logic." [28] These newly invented algebras shared the common characteristic of having extended the principles of ordinary algebra "beyond the traditional domain of pure quantity". His intention was to exhibit these algebras as "engines for the investigation of the possibilities of thought and reasoning" by showing how each could be interpreted in terms of the "abstract general idea of space." [29] This spatial interpretation is the key to Whitehead's presentation. "The result of it is", he writes, "that a treatise on Universal Algebra is also to some extent a treatise on certain generalized ideas of space." [30] These spatial interpretations provide a basis for investigating the possible applications of algebra, including their uses as 'engines' for representing the dynamic processes shaping the Universe.

In its arrangement, UA consists of seven books. The first details the basic concepts and general principles of algebraic symbolism. The second shows how these principles are realized in Boole's algebra of symbolic logic. The remaining five books are devoted to elucidating and extending Grassmann's theory of extension. The manuscript of this first volume was handed over to the publisher two years before its 1898 publication date.[31] A second volume was foreshadowed (but not published), in which he intended to focus on numerical algebras of the first species (leading to a general theory of Linear Algebras), and provide a detailed comparison of the symbolic structures of the various special algebras analyzed in both volumes.[32]

The notion of Universal Algebra had it origins in a paper of 1884, "Lectures on the Principles of Universal Algebra", by the great English algebraist James Sylvester. In his paper Sylvester spoke of the reign of Algebra the First, as a science and philosophy, ushered in by the publication of Harriot's "Artis Analyticae Praxis" in 1631 (traditional arithmetic algebra), and the reign of Algebra the Second, ushered in by Cayley's immortal "Memoir on Matrices", published in 1858 (abstract multiple algebra). From the standpoint of "Universal Algebra or the Algebra of multiple quantity", Hamilton's Quaternions "are but the simplest order of matrices viewed under a particular aspect".[33] Cayley's revolution, Sylvester argues, was rooted in the discovery that matrices, though "an organism composed of discrete parts", possessed an essential unity and wholeness, such that it "stood revealed as bona-fide multiple quantity subject to all the affections and lending itself to all the operations of ordinary numerical quantity".[34] Sylvester urged that this unification had a metaphysical significance, since it signified the transformation "whereby a multitude is capable of being regarded as an individual, or a complex as a monad." [35]

Though Sylvester's 1884 article inspired the name, and De Morgan and Hamilton were co-founders of the science of Universal Algebra, the ideas informing Whitehead's work were due mainly to Grassmann.[36] Like his British colleagues, Grassmann had arrived at the fundamental insight that a multiple sum of different quantities can be treated as an independent quantity. His entire new science, he later stated, rests on this "simple idea, which basically consists in no more than regarding multiple sums of different magnitudes (which is how extensive magnitudes appear) as autonomous magnitudes." [37] Whitehead built upon Grassmann's genius. "The greatness of my obligations in this volume to Grassmann", he states at the conclusion of his preface, "will be understood by those who have mastered his two Ausdehnungslehres. The technical development of the subject is inspired chiefly by the work of 1862, but the underlying ideas follow the work of 1844." [38] Many years later he reaffirmed that the ideas in UA "were largely founded on Hermann Grassmann's two books", and again noted that the "earlier of the books is by far the most fundamental." [39] This latter comment is of particular significance, as the Ausdehnungslehre of 1844 contains Grassmann's much-maligned philosophical introduction to mathematics and his methodological justification for the unconventional presentation of his Theory of Extension.[40]

As Albert Lewis has emphasized, the key to Grassmann's philosophy of science and of mathematics is the dialectical tension between fundamental contrasts that, unlike Hegel's notion of the dialectic, imply "no synthesis or resolution of opposites." [41] The scientific mode of presentation, according to Grassmann, consists of two interlocking series of developments. One part leads logically "from one truth to another and makes up the essential content" and the other part, arrived at intuitively, provides an over-view that "governs the process and determines the form." [42] The methodological nexus between existential interpretation and the formal system of rules has an additional ontological significance that is worth noting. Grassmann views the mind as an active body that responds to the movement of other bodies. As Michael Otte has argued, his epistemology is founded on objectively mediated activity (in contrast to Kant), and "synthesis is not foremost subjective construction, which is substantiated by the activity of the Transcendental Subject, but is representation of the real existing connections and relationships." Hence, in contrast to Euclid's "God's eye perspective" of the eternal form of space, Grassmann's fundamental entities signify "a relational mode of thinking".[43] This is exemplified in Grassmann's vision of basic entities interacting with each other (through synthesizing operations) to form more complex entities, which are then treated as the autonomous monads of a new class of manifold, or system. From this perspective, the various systems of geometry are the product of entities conforming to a particular set of laws: the ways in which the basal entities interact define the evolution of various sub-species of geometry. Thus, rather than view space as a God-given Absolute, a structure set apart from the activity that takes place within it, Grassmann's perspective interprets the spatial context as an emergent property of the interaction of the elements of the system.

Although Whitehead completed writing the five books on Grassmann before writing the book on Boole's algebra, the order in which these books appear in UA is reversed – presumably determined by the relative simplicity of the Algebra of Symbolic Logic. Whitehead's analysis of Symbolic Logic thus precedes his presentation of the Theory of Extension and serves as a gentle introduction to his key methodological notion of jointly presenting a formal calculus with its spatial interpretation. Book two thus provides a model for his subsequent style of analysis. A brief outline of the way in which Whitehead weaves together formal analysis and spatial interpretation follows. For a more technical guide to Whitehead's ideas, consult the relevant endnotes. [44]

Whitehead begins Book II by positing elements of a manifold[45], represented by the letters a, b, c, etc., and then lists the formal laws of addition and multiplication, including the general laws applicable to all algebras (such as the Commutative Law, which asserts that $x + y = y + x$, meaning that the two possible orders of synthesis produce equivalent results), and the special laws applicable only to Boole's algebra. Rigorous formal definitions are a crucial element of Whitehead's presentation. He then derives certain elementary propositions from these laws and then, before developing further algebraic formulae, shows how these laws and propositions can be interpreted in terms of concrete things and relations between things (regions of space and their intersections). For example, in Boole's non-numerical algebra, reading the formula $a + a = a$ represents the mental process of combining the same region. Adding the same element together does not reduplicate that region, since equivalent terms bring the same region before the mind, and their sum involves apprehending this identity. And while the addition of $a + b$ is equivalent in meaning to $b + a$, the meaning is not identical, since "the order of symbols is different in the two combinations and this difference of order directs different processes of thought." [46] Whitehead's spatial interpretation is notable for the phenomenological terms of its presentation: terms symbolize the mental act of apprehending a region as a concrete spatial presence; operations represent acts that combine these regions.[47] Multiplication and the various laws required and satisfied by each operation are interpreted in a similar fashion. This spatial interpretation is then developed concurrently with the algebra. For example, traditional propositions of deductive logic, such as "All a is b", can then be "conceived as stating that the region of a's is included with that of b's, the regions of spaces being correlated to classes of things." [48]

Book III, on 'Positional Manifolds', lays out the general theory of addition for algebras of the numerical genus[49] and is the key to Whitehead's interpretation of Grassmann's algebra. The first chapter of this book, 'Fundamental Propositions', follows the model adopted in his exposition of Boolean algebra. Basic definitions and the deduction of elementary propositions are combined with suggestions concerning their spatial interpretation. However, these suggestions are not framed in the explicitly psychological terms of his Boolean interpretation. This perhaps indicates an earlier phase in his progress toward an explicit demarcation between a purely axiomatic presentation of mathematical theorems and a discussion of the psychological intuitions motivating the formal propositions.[50]

In developing his basic definitions and elementary propositions, Whitehead is guided by the intuition that the elements that make up the positional manifold can be identified with "the general idea of a space of any arbitrarily assigned number of dimensions, but excluding all metrical spatial ideas." [51] In adopting a spatial model for the positional manifold, Whitehead follows Grassmann and builds his axiomatic definitions on the basis of a unit of extensive magnitude. This extensive unit is envisaged in terms of the transition of a generating element from one state to another, such that position and direction are fundamental properties of all magnitudes.[52] Adapting Clifford's image of stepping out the units that make up a line, Whitehead visualizes the production of Grassmann's basic elements as "simply the process by which any term p is transformed into the term $p + a$", where p represents a null term such that "two steps $+ a$ and $- a$ may be conceived as exactly opposed in the sense that their successive application starting from any term p leads back to that term, thus p + a – a = p." [53] Thus, Whitehead builds his axiomatic description of space on the general principles of action and movement within space.

By applying rules of change to a generating element (point), the complete system of elements constitutive of a region is defined. Every complete region "can be conceived as representing a gradually altering element which successively coincides with all the elements of the region".[54] For example, a one-dimensional region (line) "may be conceived as representing a variable element travelling through a continuous series of elements" such that, as with all linear elements in projective geometry, it finally returns to its starting point.[55] This generation of a line by a moving point is the fundamental geometrical analogy for the abstract notion of extension (in a system of the first order). Taking an element from one system and allowing it to undergo a new variety of change generates a new system. Thus, if the elements of the system of the first order are multiplied together (akin to moving a line through the series of points rectilinear to the line, so as to constitute a plane), they form another algebraic manifold (a system of the second order). By multiplying the elements of the first and second order manifolds together, a system of the third order is generated, and so on, up to any number of dimensions.[56]

One other basic characteristic of the Positional Manifold is worth noting. Each element of this manifold possesses 'intensity', which provides a numerical measure of a secondary, non-extensive property associated with the primary element. For example, it could be interpreted as a measure of the density, temperature or electrical potential of matter. A zero quantity of this scalar property signifies the absence of the element, and any other amount signifies its presence. Thus defined, the notion of intensity is fundamental to Whitehead's development of the theorems of projective geometry.[57] These theorems, he notes, "extended to any number of dimensions can be deduced as necessary consequences of the definitions of a positional manifold".[58] Whitehead's formulation of the notion of intensity in projective terms is essential to the subsequent books and is the basis of his synthesis of the Cayley-Klein approach to projective geometry and Grassmann's calculus.[59] All subsequent books are grounded in the definitions, propositions and interpretations developed in the chapters of this book.

In following this path Whitehead acknowledged that he was building upon Clifford's pioneering application of multiple-algebra to Non-Euclidian geometry, and Cox and Buchheim's independent applications of Grassmann's Calculus of Extension to Non-Euclidian Geometry. Likewise, he acknowledged that in founding metrical geometry on the definitions of his positional manifold he was extending the work of Cayley, Ball and Cox.[60] But, as he himself noted, his deduction of these various geometrical spaces from the positional manifold was facilitated because "the general idea of a pure science of extension, founded upon conventional definitions, which shall include as a special case the geometry of ordinary experience, is clearly stated in Grassmann's Ausdehnungslehre von 1844".[61]

The last book in UA discloses the practical motivations that have underpinned the entire project. Whitehead begins by stating that a “simple and useful form of the Calculus of Extension for application to physical problems is arrived at by dropping altogether the representation of the point as the primary element, and only retaining vectors,” and notes many of the formulae of physics can be “immediately translated into this notation.” [62] He concludes the volume with a final chapter, written to show that “the formulae and methods which have been developed by Hamilton and Tait for Quaternions are equally applicable to the Calculus of Extension.” The final sentence of this concluding chapter is the most direct clue to the field of application against which Whitehead is measuring the capacity of his special algebras to serve as “useful engines for the deductions of propositions”. Building upon his previous work he derives a formula representing the flow of a vector through a line of particles and then recommends that this equation “should be compared with the equations of Electromotive Force in Clerk Maxwell’s Electricity and Magnetism”. [63] In the words of Luca Gaeta, this ultimate chapter reveals the true import of UA: “it is one great design for mathematically comprehending nature.” [64] A design that performs a similar service for Maxwell’s Treatise as Newton’s Universal Arithmetic had served for his Principia.[65]

Thus, in the war between the ideas and methods of Hamilton and Grassmann, Whitehead had clearly sided with Grassmann. I have emphasized the way in which formal analysis and spatial interpretation mutually guide the development of Whitehead’s treatise. I have also suggested that Whitehead's view of Grassmann was deeply informed by the British understanding of the emerging field of Universal Algebra and its historical relation to the mathematical expression of electrical and magnetic forces. Whitehead shows how the pure science of extension can be wedded with projective geometry so as to encompass all the geometries and, following Clifford and Ball, demonstrates the close link between purely geometrical concepts and the physical concept of mass (foreshadowing his later monistic rejection of the distinction between space and the physical body occupying a space).[66] Finally, he demonstrates that for representing the flow of electric current, the Theory of Extension has advantages over Hamilton’s Quaternions. Thus Whitehead (in Gandon’s words) presents himself in spirit if not always precisely to the letter of the law, as “the true heir to Grassmann”.[67] Yet, despite its scope and practical implications, UA exerted little lasting influence on the thinking of members of the mathematical community. As we shall see, it was to be another century before UA began to find an audience.

4. The Reception of UA and its subsequent influence

Judging by the reviews, it is clear that Whitehead’s contemporaries saw his project as a call to take Grassmann seriously. G. B. Mathews, who had demonstrated his mathematical prowess by besting Whitehead for the title of Senior Wrangler in the Cambridge Tripos of 1883[68], commended its remarkable unity of design and thoroughness, and looked forward “with pleasurable anticipation” to the concluding second volume. He identified the keynote of the first volume as “Grassmann’s Extensive Calculus”, interpreted in terms of the general concept of space.[69] In a review for Mind, Hugh MacColl registered some surprise that a relatively small portion of the text was devoted to discussion of the general principles of symbolic reasoning, and that “the rest is taken up with applications of these principles, as the author understands them, to the elucidation of Grassmann's Calculus of Extension", however he commended Whitehead's "great service to science" in clarifying Grassmann's method of research that "from all accounts" was in its original exposition "extremely obscure and difficult to apply." [70] Alexander MacFarlane, an advocate of Hamilton's quaternion system of vector analysis, also praised Whitehead's work, welcoming its excellent "detailed exposition of Grassmann's system", and suggesting that the work was "likely to lead to further advances in Universal Algebra." [71] It was also well received on the Continent, especially by Louis Couturat, whose review remains one of the most extensive and comprehensive discussions of UA.[72]

Perhaps the most significant immediate impact of UA was upon Whitehead's ex-student and future collaborator, Bertrand Russell, whose efforts to assimilate Whitehead's ideas are recorded in his notes from March of 1898.[73] By April of 1898 he was determined to thoroughly revise his philosophy of mathematics, and declared to his wife, Alys, "I have finished re-reading what is important to me in Whitehead's book, and now I simply have to sit down and think and write".[74] Reflecting on this time in 1941, Russell recalled that what had greatly excited him about UA was the analysis of important branches of mathematics that were not dependent on number, and the suggestion implicit in the presentation of these "queer algebras" of "a purely formalistic treatment of pure mathematics" coupled with "an exact treatment of the conditions for the truth of formal laws (commutative, associative, etc.)".[75] Russell's Principles of Mathematics, the final outcome of his philosophical reflections from this period, still manifest traces of this influence behind the more visible presence of Cantor and Peano.[76] The publication of Russell's Principles of Mathematics, according to Whitehead, coincided with the realization that his own second volume of UA and Russell's projected second volume of Principles of Mathematics were "practically on identical topics", and their subsequent decision to combine forces to produce a joint work. They had hoped to complete this project in a year but a decade and three dense volumes later it was still incomplete.[77] Thus Principia Mathematica came into being, a monumental work that signaled a new epoch in the integration of logic and mathematics.[78]

An indication of the level of mathematical recognition achieved by UA came in 1910, when G. B. Mathews framed his entry on "Special Kinds of Algebra" in the Encyclopedia Britannica in terms of Whitehead's UA analysis.[79] In doing so he gave Grassmann's "extensive calculus" a central place in his discussion, reflecting his earlier assessment that Whitehead's "systematic development of the calculus", combined with an "abundance of illustrative applications", had deprived English mathematicians of "any excuse for ignoring Grassmann's magnificent conceptions." [80] Mathew's optimism, however, proved to be misplaced. As previously noted, in 1888 the American mathematician, Josiah Willard Gibbs had predicted that the 1890s would witness a struggle for existence between the advocates "of different methods and notations of multiple algebras, especially between the ideas of Grassmann and of Hamilton." The outcome of this struggle was a defeat for both Grassmann and Hamilton. In terms of practical application, a new contender emerged, the Gibbs-Heaviside system of vector analysis, and though "only a perverted version of the quaternion system", this system was to sweep the field.[81] A few pure mathematicians, especially those inspired by the Italian school of geometry associated with Peano, Burali-Forti and Pieri, were still promoting a Grassmannian approach - Mathews and Whitehead's later work on projective geometry bear witness to this influence – however Whitehead's Grassmannian inspired vision left a very faint imprint on the development of algebra in the new century. As Henry and Valenza recently noted, Whitehead's mathematical work, including his Treatise on Universal Algebra, "has found little audience in twentieth century mathematics." [82] However, the dream of a Universal Algebra was to be resurrected.

Garrett Birkhoff took the decisive step in defining the modern concept of Universal Algebra in 1933.[83] Although he appropriated the name from Whitehead's original text[84], he ignored the Grassmannian approach at the heart of Whitehead's work. The problem with Whitehead, G. Gratzer argued in 1968, was that although he "recognized the need for universal algebra, he has no results. The first results were published by Birkhoff in the thirties." Gratzer argues that Whitehead produced no results because, "before the thirties most of the branches of modern algebra were not developed sufficiently".[85] According to P. M. Cohn, "universal algebra as understood today goes back to the 1930s, when it emerged as a natural development of the abstract approach to algebra initiated by Emmy Noether."[86] Birkhoff argues that Noether's influence was mediated by the text that defined the course of modern algebra, B. L. van der Waerden's Moderne Algebra.[87] Noether and her school looked to a tradition inspired by the approach of Dedekind, rather than Grassmann. The numerical inspiration behind Birkhoff's Dedekind-Noether lineage can be contrasted with the geometrical inspiration of Mathew's Grassmann-Whitehead lineage.[88] Emmy Noether's bias was expressed, according to Gian-Carlo Rota, in her passionate hatred of classical invariant theory and anything that hinted of the "mystifying concoctions of that crackpot Grassmann." Thus Noether's influence contributed to what, Rota argues, was "the mathematical tragedy of this [the twentieth] century": the failure to learn from Grassmann.[89]

The present day rehabilitation of Grassmann's reputation had its roots in the work of Elie Cartan, who had drawn upon Grassmann in formulating his theory of differential forms.[90] His work was little understood until the 1930s, when it was rediscovered in the light of the modern understanding of linear algebra.[91] This rehabilitation gathered further momentum in 1948, when the Bourbaki group published their rigorous, axiomatically formulated version of the exterior calculus.[92] However, it was a flawed presentation of Grassmann's ideas, stripped of essential insights that Grassmann had arrived at by visualizing his calculus of extension in geometrical terms. The problem with Bourbaki's set theoretical presentation, as René Thom argued, was that their algebraic focus on logical structure and proofs failed to acknowledge and benefit from the intuitive sensibilities that are an integral part of mathematical insight.[93] Hence, important results were neglected and obscured by the Bourbaki presentation. These failings were in part recognized in 1985, when Rota's group published their seminal article, "The exterior calculus of invariant theory".[94]

Rota's work on Grassmann reflected a new algebraic trend, away from van der Waerden's emphasis on an axiomatic approach to deductive systems, toward a focus on relational structures, and themes and techniques that reflect areas of mathematics and applications opened up by computers.[95] Rota and his co-workers acknowledged the importance of this factor in 1988, when they noted that their "modern version of Grassmann algebra" had, in the last few years, captured attention "because of its wide applicability to 'effective projective geometry' . . . [and its utilization as a tool for] computer vision." [96]

Under the headline "Hermann Grassmann was right", Ian Stewart outlined Rota's findings in an article in *Nature*, suggesting that by combing the conceptual insights of Grassmann with Bourbaki's axiomatic rigor such that "the notation reflect the concepts", Rota had earned the title of "the Edgar Allen Poe of mathematics".[97] Thus, after almost a century of neglect, a revival of interest in Grassmann's work took place.[98] During the 1990s the first English translations of the *Ausdehnungslehres* were published[99] and prominent mathematicians such as William Lawvere, best known for his important contributions to Category Theory, began urging mathematicians to take another look at Grassmann.[100]

As a consequence of the Grassmann revival, a number of mathematicians have drawn attention to Whitehead's UA. Barnabei, Brini and Rota initiated this interest when they argued in their 1985 article that Whitehead was one of the few mathematicians that had understood Grassmann. Though their remarks focused on Whitehead's technical presentation of Grassmann's theory of extension, Whitehead's broader vision of a Universal Algebra has also attracted interest. John Browne notes that Grassmann's work was one of the foundation stones upon which Whitehead had "hoped to build an algebraic theory which united the several important and new mathematical systems which emerged during the nineteenth century", and suggests that, apart from the translations of the original texts, Whitehead's UA is "probably the best and most complete exposition on the *Ausdehnungslehre* in English".[101] David Hestenes, whose work has been a primary factor in promoting the renewed interest in the geometrical approach to algebra inspired by the Grassmann-Clifford-Whitehead tradition, has argued, "Grassmann and Whitehead were just one step away from a mathematical system that truly deserves to be regarded as a Universal Geometric Algebra".[102] "I am still amazed", he recently commented, "at how little attention was paid to [Whitehead's] work on "Grassmann's algebra of extension", and that "when I finally got around to looking carefully at *Universal Algebra* I found that I had independently arrived at similar conclusions on most of the topics." Hestenes also comments that Whitehead was the first to discover "the very powerful idea that I call outermorphism", although his discovery seems to have remained unnoticed.[103]

As Rota noted, the interest in Grassmann has been fostered by the advantages of his algebra as a tool for applying projective geometry in the development of computer vision. This concern with practical applications lies behind Stephen Blake's recent defense of Whitehead's approach to algebra. In *Whitehead's Geometric Algebra*, Blake argues that Whitehead's interpretation of Grassmann is superior to the approach advocated by Barnabei, Brini and Rota in their 1985 article. This superiority has its roots in Whitehead's approach to projective geometry, and his recognition that the various kinds of products differentiated by Grassmann are variations of a single product.[104] Blake's work points towards some of the current technical challenges motivating the renewed consideration of Whitehead's work.

5. The Metaphysical significance of UA

At the heart of Whitehead's presentation of Grassmann's Theory of Extension was a vector notion of transition. UA was informed by a model of the cosmos in which Newton's autonomous atoms were replaced by a relational notion of linear strains. Crucial to this move was the notion that strains and stresses be viewed as generative forces, organically binding together and thereby bringing into being, larger unities.[105] Whitehead's subsequent elaboration of this idea can be traced through his later scientific and metaphysical works, culminating in the notoriously difficult "Theory of Extension" presented in Part IV of PR. This is not the place to discuss these later developments in Whitehead's systems of thought, however, I do wish to make a final comment on a neglected aspect of the Grassmann-Clifford-Whitehead relation.

Lowe, in seeking to discover the origins of Whitehead's relational views, asked Russell if Whitehead, in the early years of their collaboration, had also embraced the absolute theory of space. Russell responded "No", and added, "I think he [Whitehead] was born a relativist." [106] I have suggested that Grassmann's philosophy was a source of inspiration for Whitehead's relational orientation. Clifford's metaphysical writings dramatically illustrated the cosmological implications of this standpoint.[107] Whiteheadian scholars have paid little attention to either figure. After UA, both become largely invisible presences in Whitehead's works. With Clifford, this issue of invisibility was exacerbated by his early death. Many of his most exciting ideas were still in a tentative form – the seeds of future works. One of these ideas was that the elementary entities of the cosmos, preceding both matter and space, were feelings. This suggestion is outlined in two remarkable articles that together can be read as a charter for the development of a geometrical model of the cosmos, based purely on the notion of extension.[108]

According to Clifford, the world portrayed by science, in all its material objectivity, had its factual basis in the subjective domain of human thoughts and feelings. He suggested that the objective facts of science are socially constituted sensibilities and, basing his views upon "recent advances in the theory of perception" [109], argued that the universe "consists entirely of mind-stuff." Particles of mind-stuff make up the ether of Clifford's universe. "These elements of feeling", he writes, "have relations of nextness or contiguity in space, as exemplified by the sight-perceptions of contiguous points; and relations of succession in time which are exemplified by all perceptions." The mathematician, Clifford argued, must build a model of the universe from these two empirical relations, helped by recent work in geometry indicating that distance and quantity can be derived from an analysis based purely on the notion of position, and theories of space curvature that hint "at a possibility of describing matter and motion in terms of extension only."[110] In other words, Whitehead's notion of equating the vector forces of physics with subjective feeling probably had its roots in Clifford's work. From this perspective, Whitehead's UA can be viewed as a first step towards Clifford's vision of giving rigorous mathematical expression to the basic elements of his metaphysics of feeling.[111]

6. Conclusion

In my presentation of UA I have emphasized the part Grassmann and his British interpreters played in the formation of Whitehead's vision, and focused on three interrelated aspects of Whitehead's work:

- grounding the development of a formal axiomatic system in a phenomenological reflection on the subjective process of spatial reasoning
- defining basic algebraic entities in terms of dynamic relational processes
- developing Grassmann's theory of extension as a practical tool for representing Maxwell's equations.

I have also hinted that, while UA exerted little influence on the wider mathematical community, it did represent the first obscure expression of a philosophical vision that was to become prominent in Whitehead's later writing.

Throughout the chapter I have emphasized the leading role played by the notion of space. Galileo believed that Euclid had disclosed the spatial language employed by the Great Designer in the creation of the Universe. Inspired by the Grassmannian approach of Clifford and his followers, Whitehead broke free of this stifling heritage and embarked on a mathematical adventure, the ultimate goal of which was to create a new model of the Universe. In so doing, he adopted a relational view of space, in which ontological priority is assigned to relations between events, rather than to absolute, apriori contexts of action. This radical shift in philosophical perspective was couched in an inaccessible mathematical language. The result was misunderstanding and ill-founded criticism.

In the only book review he ever published,[112] the genial Whitehead savagely skewered Hastings Berkeley's Mysticism in Modern Mathematics [113]. However, having pilloried Berkeley's understanding of recent developments in mathematics, Whitehead did concur with some of Berkeley's criticisms of UA. Henry and Valenza have suggested that Whitehead's response to Berkeley signals Whitehead's repudiation of his early philosophical standpoint (a "tentative formalism"). Further, they argue that he then adopts (with Russell) the "logicism of Principia Mathematica", and then, after abandoning this position, is left floundering "with no explicit coherent philosophy of mathematics." [114] I contend that Whitehead's early and late testimony to the importance he attached to the Ausdehnungslehre of 1844 suggest an alternative possibility. Namely, that the 'philosophical' approach of the first edition, though incomprehensible to most of Whitehead's mathematical colleagues, struck a chord with Whitehead. This chord was amplified through the writings of Clifford and inspired the monumental production that was to become the mathematical overture to Whitehead's life work.

Notes

[1] Galileo Galilei, Il Saggiatore (The Assayer), 1623, quoted in Sussman & Wisdom 2002, p. 2.
[2] Whitehead, in Schilpp 1951, pp. 659-670.
[3] In 1903 the Treatise on Universal Algebra led to Whitehead's election to the Royal Society. The book was also well received on the Continent, according to Russell, who in August 1900 had accompanied Whitehead to the Paris Congress of Philosophy; as he later told G. E. Moore "all the foreigners who knew Mathematics had read and admired his book, and were delighted to meet him." Grattan-Guinness 2002, p. 432.
[4] Keeton 2004, pp. 32-33.
[5] Whitehead, letter of 1912, in Lowe 1975, p. 86.
[6] Harman 1998, p.6.
[7] Whitehead 1938, pp.136-138.
[8] Although Maxwell died in 1879, his work had inspired a new generation of mathematicians and physicists. Through the eighties, leading 'Maxwellian' physicists such as Oliver Heaviside and J. J. Thomson, continued to clarify Maxwell's notion of an electromagnetic field, and to simplify the equations he used to describe electrical charges, magnetic dipoles and the relationship between them both. In 1888, Maxwell's field theory of electromagnetic waves received empirical confirmation through Heinrich Hertz's production and detection of electromagnetic waves. These technical advances took place in the context of simmering philosophical controversies over the true method of science, the basis of mathematical and scientific concepts used to define dynamic systems, and the nature of the actual realities denoted by those concepts.
[9] Whittaker, 1948 p. 281.
[10] These developments are described in chapters 7 and 8 of Harman 1998.
[11] Klein spelt out his vision in his "Erlanger program" of 1872. He drew upon the study of the invariant properties of groups, to show that the different classes of transformations defined different branches of geometry.
[12] An intuitive sense of the contrast between projective and metrical geometry can be gained by imagining mapping your position in a city by using a photograph, such that the perspective captured in the photograph exactly specifies the position from which it was taken, compared to locating your position by measuring the distance of your position from a triangle of landmarks.
[13] Lowe 1985, p.151.
[14] James Beichler details the influence of Clifford on the key British mathematicians that guided Whitehead's approach to Hamilton, Grassmann and Maxwell (Beichler 1996). Joan Richards describes the wider context of these developments in her book, Mathematical Visions: The Pursuit of Geometry in Victorian England (1988).
[15] Hamilton had initially been enthusiastic about Grassmann but had become increasingly critical, noting certain resemblances between their notions of vectors (directed lines) but failing to appreciate that Grassmann's concepts and method could be extended to encompass his theory of quaternions (cf. Crowe 1967, pp. 86-87. Klein valued the Ausdenungslehre of 1862, but regarded the 1844 Ausdehnungslehre as "virtually unreadable". Klein's relation to Grassmann is analysed by Rowe, in Schubring 1996, pp.139-144.
[16] Clifford 1882, p. 266.
[17] Clifford also emphasized the importance of R. S. Ball's work, another figure that Whitehead later acknowledges in UA. Clifford thus left behind the contours of a project that the British 'Grassmann-Clifford School' developed. For a contemporary mathematical colleague's estimation of Grassmann's impact on Clifford's thought, see Smith's "Introduction" in Clifford (1882, pp. lv-lxx).
[18] Malonek 2004, p. 7. Malonek points out that Crowe's in many respects admirable History of Vector Analysis (1967) follows the tendency of minimizing Clifford's contribution (perhaps for the same reason that he chooses not to deal with Whitehead).
[19] Richards 1988, p. 131. Clifford achieved notoriety in his day primarily because of his outspoken views on religion. MacFarlane commented that Clifford "could not write on any religious question without using language which was offensive even to his friends." MacFarlane 1916, p. 57 He challenged doctrines advanced by Hamilton, Maxwell, and Tait, as outlined by MacFarlane (1916). Tait, in particular, was a recipient of Clifford's barbs, as a result of his efforts to advance religious conclusions via scientific argument. Tait repaid the favor by accusing Clifford of plagiarism – after Clifford was dead. Clifford's

outraged friends responded on his behalf. See Clifford's review of Tait in the "The Unseen Universe" (Clifford 1879a), and Tait's review, "Clifford's Exact Sciences", in Nature 32, 11 June 1885 (with responses on the 18 and 25 June, and the 2 July in the same volume). For bibliographic details on this exchange, see Chisholm, 2002.

[20] Richards 1988, pp. 109-113.

[21] The notion that Clifford or Geometric Algebra can serve as the basis of a unified language for mathematics is advanced in Hestenes 1986, and has been taken up by other enthusiasts. A historical perspective of Clifford's mathematical legacy is presented in Chisholm 2002; Diek and Kantowski 2005; and Lounesto 2001.

[22] Russell, for example, recalls his tutor giving him Clifford's The Common Sense of the Exact Sciences (in his introduction to Clifford 1885, p. v) in order to introduce him to mathematics. Whitehead refers to Clifford's Elements of Dynamics (vol. 1 1878, vol. 2 1886) in his introduction to UA (without providing bibliographic details, cf. UA p. 25), in a manner that reflects its wide circulation amongst mathematicians. Much of Clifford's work was being published posthumously during the eighties, and was the occasion of considerable interest (see previous note). Whitehead makes numerous references to Clifford in the text of UA and notes that he intends to deal specifically with Clifford's algebra in volume 2 (cf. UA, p. 370).

[23] Cox 1882a, p. 10 A brief outline of Cox's paper is provided in Cox 1882b.

[24] Buchheim 1884, p.88. The mathematical concept of 'screw' – the simultaneous rotation around and proportional translation along an axis, like the thread of a screw – was fundamental to Clifford's theory of the ether. For a short account of Buchheim's career, detailing the influence of Clifford and Klein, and the series of publications that Buchheim produced in the eighties "promoting Grassmann's methods", see Tattersall 2005.

[25] Whitehead notes his particular debt to Buchheim and Cox "as the mathematicians whose writings have chiefly aided me in the development of the Calculus of Extension" in UA, pp. x, 248, 346, 370 & 575.

[26] Announcement from the Cambridge University Reporter of June 10, 1887, cited in Lowe 1985 p.150.

[27] Letter from Gibbs to Thomas Craig, quoted in Crowe 1967, p.182.

[28] UA, p. v.

[29] UA, pp. viii & v.

[30] UA, pp. 31-32.

[31] I base these dates on the preface, dated December 1897, and a concluding note dealing with some recent publications that he was unable to deal with, due to the book already having been "nearly two years in press." UA, p. 573.

[32] Whitehead gives some indication that this second volume would include detailed analysis of Cayley's matrices, Hamilton's Quaternions, Clifford's Biquaternions, and the vector algebras of Heaviside and Gibbs, see UA pp. v, 32, 370 & 573. Whitehead later came to believe that his goal of analysing the symbolic structure of these various algebras had been subsumed within the aims of PM, see Whitehead, in Schilpp 1941, p. 10.

[33] Sylvester 1884, p. 271.

[34] Sylvester 1884, pp. 270-271. For a modern perspective on the historical importance of Sylvester's work, see Mc Intyre 1991.

[35] Sylvester, quoted in Mc Intyre 1991, p. 574.

[36] For a historical perspective on the 19th century British approach to algebra and their notion of a Universal Algebra see Novy 1973 & 1976.

[37] Grassmann, 1862, p. xiv.

[38] UA, p. x.

[39] Whitehead, in Schilpp 1951, p. 9.

[40] Hamilton, referring to the 1844 version of the Ausdehnungslehre, wondered whether a pipe of opium would unlock its secrets (Crowe 1967, p. 86); Felix Klein described it as "extraordinarily obscure" (Klein 1939, p.20); Morris Kline summed up the typical response by suggesting that Grassmann had "shrouded the ideas [of 1844] with mystic doctrines", producing a work that was "vague and unreadable" (Kline 1972, p.782). Crowe (1967) provides a detailed accounts of the reception of the Ausdehnungslehre (1844), pp. 77-88.

[41] Lewis 2004, p. 21.

[42] Ausdehnungslehre, 1844, p. 30.

[43] Otte 1989, pp. 25-27.

[44] Gaeta 2002, provides a general discussion of the development of UA. Jaques Riche provides a more technical introduction (Riche 2004). Quine 1951, provides a good introduction to Whitehead's discussion of symbolic logic. There is no text in English, that I am aware of, that examines Whitehead's presentation of the Theory of Extension, though his work is touched upon in several texts dealing with the development of Geometric Algebra (see section 4). Perhaps the best introduction to Geometric Algebra for the non-specialist, though not directly addressing Whitehead's work, is Hestenes 1986b.
[45] Whitehead's concept of 'manifold', loosely based on the ideas of Riemann and Grassmann, is defined as the aggregate of all the modes in which something possesses a particular property. For example, the coordinate axes defining an empty space specify all the modes that points in that space can assume. The points are the elements of this spatial manifold. Similarly, Whitehead speaks of the "manifold of musical notes conceived as representing every note so far as it differs in pitch and quality from every other note." UA p. 15.
[46] UA p. 6. Whitehead's distinction between identity and equivalence is discussed in Mays 1977 (chapt. 3) and Quine 1951.
[47] The following passage exemplifies Whitehead's interpretive stance: "Let the elements of this algebraic manifold [a,b,c, etc.] be regions in space, each region not being necessarily a continuous portion of space. Let any term symbolize the mental act of determining and apprehending the region which it represents. Terms are equivalent when they place the same region before the mind for apprehension.
Let the operation of addition be conceived as the act of apprehending in the mind the complete region which comprises and is formed by all the regions represented by the terms added. Thus in addition the symbols represent firstly the act of the mind in apprehending the component regions represented by the added terms and then its act in apprehending complete region. This last act of apprehension determines the region which the resultant term represents." UA p. 38-39.
[48] UA, p. 99.
[49] Positional manifolds are defined by the common property of permitting addition of the numerical type, such that, in contrast to Boole's non-numerical algebra of logic, a and a + a are not equivalent; instead, in numerical algebras, $a + a = 2a$.
[50] This distinction becomes explicit in Whitehead 1906, p. 17.
[51] UA, p. 30.
[52] See Grassmann's (1844) discussion, pp. 45-46. Note that there is some debate over whether the basic entity of Grassmann's algebra should be viewed as a point, a vector, or as a unit of change. A similar ambiguity arises in the course of Whitehead's discussion. See Lewis 2004, fn. 14, p. 32
[53] This provides Whitehead with a very simple visual image of the commutative law (steps can be taken in any order), and the associative law (any number of steps can be replaced by one definite resultant step). UA p. 25.
[54] UA, p. 127.
[55] UA, p. 127.
[56] UA, p. 27.
[57] Maxwell, drawing upon Gauss, had used the term 'geometry of position' in 1871 for the projective geometry of Lazare Carnot and Michel Chasles. Gauss had been inspired by "Leibniz's remarks on the need to formulate geometric algorithms to express geometric location (situs)." (Harman 1998, p. 154.) Whitehead's formulation of the notion of a positional manifold can be viewed as a contribution to the efforts of various Grassmannians to explicate the broader algebraic implications of projective geometry. See, for example, Mr E. Lasker's "Essay on the Geometrical Calculus", in which he attempts to demonstrate that "Grassmann's Ausdehnungslehre is a shape into which projective geometry or modern algebra may be thrown; that it is coextensive with these two branches of mathematics" (Lasker 1896-7, pp. 217-8).
[58] UA, p. 132.
[59] Sébastian Gandon argues that Whitehead's reformulation and generalization of Grassmann's notion of intensity in projective terms enables him to demonstrate the great scope of Grassmann's theory. It enables him to formulate the Absolutes (invariants) of the traditional metrical (Euclidian and Non-Euclidian) geometries in terms of various 'laws of intensity' and demonstrate that they are sub-species of projective geometry. See Gandon 2005; also Gaeta 2002, p. 131.
[60] See the note UA, p. 370, also Gandon 2005.
[61] UA, p. 370.

[62] UA, p. 548.
[63] UA, p. 573.
[64] My translation, Gaeta 2002, p. 132.
[65] Newton had written his classic text Universal Arithmetic to help substantiate and advance the theory of equations (an algebraic interpretation of Euclidian proofs) underpinning his Philosophiae Naturalis Principia Mathematica.
[66] Whitehead 1906.
[67] My translation, Gandon 2005, p. 6.
[68] Lowe 1985, pp.102-103.
[69] Mathews 1898, p.385.
[70] MacColl 1899, p.108.
[71] MacFarlane 1899, p.328.
[72] Couturat 1900.
[73] The editors of Russell's papers from that period describe 1898 as a "watershed" in Russell's philosophy of mathematics, brought about by fresh ideas, the chief source of which was "Whitehead's Universal Algebra (1898), which 'greatly excited' Russell when he read it in March 1898" (Russell 1990, p. xxv).
[74] Russell, in a letter to Alys, 1st April, 1898 (Russell, 1990, p.155).
[75] Russell, in a letter dated June 18, 1941, in response to a query from Lowe, (quoted in Lowe 1962, p. 144).
[76] It is interesting to note that in this work he affirmed that the essential principles of Universal Algebra had to be sought inductively, through an examination of the various species of Algebra, and suggested that his philosophical writings on this theme complemented Whitehead's mathematical inquiries in UA. Russell [1903] 1996, pp. 376-377.
[77] Whitehead 1951, p.10. Russell announces their collaboration in the 1902 preface to his Principles of Mathematics (Russell [1903] 1996). Three volumes of Principia Mathematica were published, the fourth volume, for which Whitehead was responsible, was never completed. For details, see Grattan-Guinness 2000.
[78] Consideration of PM raises many questions concerning the role of UA in the early development and later evolution of Whitehead's mathematical and philosophical thought. In discussing the creation and subsequent influence of UA my narrative has focused on the Grassmannian legacy. I have noted that deep philosophical questions concerning the relationship between logic and mathematics were of interest to both Grassmann and Whitehead. UA was clearly a significant factor in the development of Whitehead's and Russell's approach to the philosophy of mathematics, and arguably this chapter should consider the broader philosophical influence of UA. I have not attempted to do so, but an indication of the complexity of these philosophical patterns of influence are suggested by Juliet Floyd's analysis of the way in which UA influenced Wittgenstein's treatment of arithmetic in the Tractatus. See Floyd 2001, pp.152-155, and Floyd, 2005. For a heroic attempt to provide an overview of the complex tapestry of mathematical and philosophical influences that formed the context of the collaboration between Whitehead and Russell, and an evaluation of the subsequent influence of PM, see Grattan-Guinness 2000.
[79] When listing the authorities on the subject, Mathews notes the "very comprehensive" nature of Universal Algebra and suggests that his approach to the subject is "in many ways indebted" to this work (Mathews 1910).
[80] Mathews1898, p.385.
[81] cf. Crowe 1967, p. 215.
[82] Henry and Valenza 1993, p.157.
[83] Burris and Sankappanavar 1981, p.25. This was the first in a series of influential articles, Birkhoff 1933, 1935 & 1944.
[84] Birkhoff 1976, [The rise of modern algebra to 1936, Graduate Studies, no 13, Texas Tech. University], cited in Fearnley-Sander 1982, p.162.
[85] Gratzer 1979, p. vii.
[86] Cohn 1981, p. xv.
[87] Birkhoff suggests that "it would be hard to overestimate the influence which van der Waerden's concept of modern algebra, as popularized by subsequent authors, has had on mathematics", but makes the point that he and his followers "ignored entirely Boolean algebra". He acknowledged, in 1946, that Noether's school "had developed many of the most important ideas of Universal Algebra" (Birkhoff 1969, p.431).

[88] See Fearnley-Sander 1982, pp.162-163.
[89] Rota 1997, pp.46-48, 232-233. Fearnley-Sander argues that Birkhoff's approach to algebra is indebted to the work of van der Waerden's patron, Emmy Noether, and that the neglect of Grassmann reflects Birkhoff's indebtedness to the Noether lineage (Fearnley-Sander 1982, pp.162-3) Rota paints a more complex picture, in which Birkhoff is identified with Philip Hall and a tradition that looks back to Boole and Peirce, while van der Waerden and Noether look back to Kronecker and Dedekind. According to Rota the German-French approach to algebra was dominant for most of the twentieth century, promoted more recently by the likes of Emil Artin. Birkhoff, on the other hand, in opposition to Artin, is implicitly grouped with those more open to a re-evaluation of Grassmann (see Rota 1997, pp.53-54).
[90] Stewart 1986, p.17.
[91] Dieudonné 1979, p.11.
[92] Rota suggests that all the mathematicians of his generation came to exterior calculus via Bourbaki (Rota 1997:50).
[93] Thom 1971, pp. 695-696
[94] Barnabei, Brini & Rota 1985, pp.120-122.
[95] See Birkhoff 1973, p.779.
[96] Grosshans, Rota & Stein 1988.
[97] Stewart 1986, p.17.
[98] See the collection of articles in Schubring 1996.
[99] Translated by Lloyd Kannenberg, cf. Grassmann 1995 and Grassmann 2000. Kannenberg has also produced the first English translation of Peano's 1888 text on Grassmann (Peano 2000).
[100] See Lawvere in Schubring 1996, pp.255-264, and most recently Lawvere 2005, pp.99-105
[101] Browne 2001, p. 5.
[102] Hestenes 1988, p. 1.
[103] Hestenes, personal communication, 28 April, 2005.
[104] Blake 2005, pp. ix-x.
[105] Douglas Bendall has developed this theme, connecting Whitehead's early relational view of the material world (Whitehead, 1906) with later works. He quotes Whitehead: "At some stage in our account of stress we are driven to the concept of any extended quantity of material as a single unity whose nature is partly explicable in terms of its surface stress." (Whitehead 1919, p. 3) Bendall 1973, p. 21.
[106] Interview with Russell in 1965, quoted in Lowe 1975, p. 91.
[107] Although Whitehead acknowledges Clifford as one of the main sources of inspiration for his work on non-Euclidian geometry (UA: x), the nature of broader influences remains speculative, since Whitehead (while acknowledging their importance) quite deliberately avoids making comments on psychological and metaphysical subjects until many years after the publication of UA.
[108] See Clifford 1879a and 1879b. Clifford's approach owed much to the great German polymath, Hermann von Helmholtz, as discussed in Farwell and Knee 1990. In Russell's Essay on the Foundations of Geometry, Clifford is classified as a "naïve realist" with regard to his views on space, in that he shares the empiricist approach of Riemann and Helmholtz (cf. Russell [1897] 1956, p. 97). Grassmann was also an empiricist with respect to the mathematical form of space. Helmholtz promoted the ideas of Grassmann, as discussed in Darrigol 2003. Grassmann had first been exposed to the Romantic view of Nature through his father, a disciple of Schelling. According to Schelling: "Nature is visible mind and mind is invisible nature" (Heuser 1996: 50). Nature has an inner being that is self-productive and self-organizing. The mind's own productive activity was the inner manifestation of the same forces that constitute nature. According to Schelling, these forces did not interact within space, rather, they were generative of space: their expansions and contractions give rise to matter and space, feeding upon themselves to create process of higher orders in a way that pre-figures Grassmann (cf. Heuser 1996, p.53). These themes were further reinforced by Grassmann's reading of Schleiermacher (cf. Lewis 1977).
[109] Clifford 1879b, pp.87-88.
[110] Clifford 1879a, pp. 244-245.
[111] This charter is still of interest to contemporary physicists, as suggested recently by Meschini et. al., when they noted that "Clifford pursued the conviction that matter and geometry do not lie at the bottom of things, but that, on the contrary, they are themselves an aspect of deeper entities – for him, feelings – and their non-geometric relations, contiguity and succession", and relate his work to their investigation of the

pre-geometrical entities and interactions that give rise to the "the quantum structure of space and time." Meschini et. al. 2004, pp. 28-29.
[112] Whitehead 1911.
[113] Berkeley 1910.
[114] Henry and Valenza, 1993, p.167.

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