

Theories of Elasticity and Fluid Dynamics Established from New Dynamic Hypotheses

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Abstract. The theories of elasticity and fluid dynamics are two basic theories in continuum mechanics. Traditional continuum mechanics regards the motion of material element forming a continuum as the translational motion of rigid body, and the theorem of moment of momentum about an axis passing through the centroid of material element is added to describe the dynamics of the material elements of a continuum. Under the cognition, the gradient of displacement in the theory of elasticity is divided into symmetric and antisymmetric components known as the strain tensor and rotation tensor, respectively, and the gradient of velocity in the theory of fluid dynamics is divided into symmetric and antisymmetric components known as the strain rate tensor and rotation rate tensor. However, there are several questions in the theories of elasticity and fluid dynamics developed from above assumption: 1. The rigid body rotation termed in geometric relationships conflicts with the assumption of translational motion of material elements; 2. The constraint of compatibility conditions on displacement field ($\nabla \times (\nabla \mathbf{u} + \mathbf{u} \nabla) \times \nabla = 0$) is different from the property of displacement field ($\nabla \times (\nabla \mathbf{u}) = 0$), which may cause that the solutions to elasticity problems solved with displacement method are different from the solutions to elasticity problems solved with other methods; 3. The decomposition of gradient of displacement and velocity is inconsistent with the decomposition of vector fields which are often decomposed into divergence field and curl field; 4. The calculation of stress on an inclined plane is equivalent to denying the necessity of body force and inertial force in Cauchy's equation of motion; 5. Stokes hypothesis is required in the theory of fluid dynamics to guarantee that the pressure derived from the fluid's constitutive relationship is equal to the thermodynamic pressure. The study re-establishes the theories of elasticity and fluid dynamics based on new dynamic hypotheses to replace current

imperfect theories. In the new theories, it is assumed that the theorem of momentum is the only dynamic law that a continuum obeys, which describes the translational motion of deformable media. Under the hypothesis, a new dynamic equilibrium equation is proposed, stress tensor is an asymmetric tensor which can be described with the gradient of a vector, and strain and strain rate tensors are the gradient of displacement and velocity, respectively. The properties of force field in continuum can be obtained by the symmetry of the stress tensor and strain (rate) tensor. Based on the new dynamic hypotheses, the constitutive relationships for elastomer and Newtonian fluid are modified accordingly. The traditionally defined wave equation and Navier-Stokes equation are obtained from the new theories of elasticity and fluid dynamics, respectively. The new theory of elasticity does not require compatibility conditions, and the new theory of fluid dynamics does not require Stokes hypothesis. In the new theories, the curl of displacement and flow fields in continuums is no longer related to rigid body rotation.

Keywords: Classical continuum mechanics; Stress state; Strain state; Elasticity; Fluid dynamics

1. Introduction

Continuum mechanics studies the motion, deformation and failure of deformed media such as fluids and solids under the continuum hypothesis, where real fluids and solids are considered to be perfectly continuous and are paid no attention to their molecular structure [1, 2, 3]. Continuum mechanics is the basis and framework of engineering science. With the continuous development of engineering and technology, continuum mechanics has been widely applied in aerospace [4, 5], information technology [6, 7], biomedical engineering [8, 9], micro/nano technology [10, 11, 12] and other fields. At the same time, the application of continuum mechanics in these fields promotes its own development. The development of theoretical research on the mechanical properties of continuous media is of great significance for the development of human civilization.

At present, the community of mechanics generally believes that continuum mechanics is a branch of classical mechanics, and the motion of material elements constituting a continuum is the motion of particles, which can be described by Newton's three laws of motion or other mechanical principles related to and equivalent to them [1-3, 13-15]. Continuum hypothesis allows for the description of internal force acting on every given surface element in the form of a field and the use of powerful methods of calculus to describe the equilibrium of a free body with an infinitesimal volume in a continuum [1, 3]. In order to conveniently describe the force on the bounding surface of a free body and the equilibrium of the free body whose volume goes to zero under resultant force, the stress tensor is introduced into continuum mechanics [1-3, 13]. The stress tensor is shown to be a second-order symmetric tensor in classical continuum mechanics because the motion of nonzero-volume elements constituting continua is treated as the motion of particles, and

the theorem of moment of momentum is applicable to describing the dynamics of material elements constituting a continuum [13, 14, 16].

In a continuum, strain or strain rate is what causes stress. For instance, strain causes stress to build up in an elastomer, and strain and strain rate cause stress to build up in a fluid. In continuum mechanics, the strain tensor and strain rate tensor are introduced to conveniently explain the constitutive relation of a variety of continua. The strain tensor and strain rate tensor are second-order symmetric tensors since the stress tensor is a second-order symmetric tensor [13-16]. In the theory of elasticity, the symmetrical portion of the gradient of the displacement field is used to define the strain tensor, and geometric equations link the six strain components to the three displacements [15]. When the distribution of stress or strain in an elastomer is known, the strain tensor is thought to satisfy specific integrability conditions since strain only contains a portion of the information about the displacement field. As a result, the theory of elasticity introduces the compatibility requirements. In the theory of fluid dynamics, the strain rate is the symmetric part of the gradient of velocity field [2]. Similar to the velocity field, the strain rate only contains some information about the velocity field, so the strain rate should also meet certain integrable conditions. However, as fluid dynamics focuses on the distribution of flow fields, fluid dynamics problems are often solved by solving Navier-Stokes equation. The integrable conditions are unnecessary for fluid dynamics researchers to take into account.

The application of continuum mechanics, which is based on traditional dynamic assumptions has greatly promoted human understanding of the motion, deformation, and failure of continuum. Practice has proved that the achievements, such as the wave equation

and Navier-Stokes equation, are undoubtedly correct. However, we can see that continuum mechanics still contains some inconsistencies. Taking transverse waves as an example, the equation for transverse waves can be derived from the theory of elasticity, however the theory of elasticity has difficulties in explaining transverse waves. Transverse wave is defined as the motion of all points on a wave oscillate along paths at right angles to the direction of the wave's advance. In the theory of elasticity, Transverse waves are the superposition of the deformation and the rigid body rotation. Under the law of equivalence of shear stress derived from the conservation of moment of momentum, the element cannot rotate, which runs counter to the physical justification for the antisymmetric portion of the displacement gradient. At the same time, in fluid dynamics, there are difficulties in distinguishing between laminar flow and turbulence, and Stokes hypothesis is required when the pressure is determined from constitutive relation of fluid even though the cause of the viscous force has been identified by shear deformation. The study believes that the introduction of unsuitable dynamic hypothesis, the theorem of moment of momentum, is what causes the paradoxes in continuum mechanics. By introducing the theorem of moment of momentum about an axis running through the centroid of the material element the motion of material elements comprising a continuum is actually treated as rigid body translational motion in dynamics. Since continuum is a deformable body, continuum mechanics should develop a particle equilibrium equation according to the deformable body's properties.

In order to more accurately characterize the motion and deformation of elastomers and fluids, the study intends to re-establish the theories of elasticity and fluid dynamics based on new dynamic hypothesis. For the new theories of elasticity and fluid dynamics,

the compatibility conditions and Stokes hypothesis are no longer necessary, respectively. The displacement and flow fields in continua are no longer related to rigid body rotation. The stress on an inclined plane is the directional derivative of a vector, which is named as the stress vector. The rest of the study is organized as follows. Firstly, the paradoxes in classical continuum mechanics are pointed out: 1. The constraint of deformation coordination on displacement field is different from the property of displacement field; 2. The constraint of the conservation of moment of momentum on shear stress is dependent on the position where the stress is expanded; 3. The calculation of stress on an inclined plane denies the necessity of body force and inertial force in Cauchy's equation of motion. Then, Newton's second law, which describes the dynamics of discrete particles, is extended to differential form. By examining the relationship between a vector field's characteristics and its gradient, the validity of the new differential form of Newton's second law is demonstrated, and that the stress tensor may be described using a vector's gradient is obtained. Finally, by correctly altering the constitutive relationships of the fluid and the elastomer, the theories of elasticity and fluid dynamics are constructed. The classic definition of the wave equation and the Navier-Stokes equation are obtained by expressing the equation of motion with displacement and velocity. The validity of the new theory of elasticity is verified by solving the problem of a wedge subjected to dead-weight and liquid pressure as an example.

2. Paradoxes in classical continuum mechanics

A proper theory has no contradictory statements and the conclusions drawn from the theory often do not change depending on the coordinate choice. In this section, some paradoxes in classical continuum mechanics are illustrated.

2.1 The displacement field property of elastomer

Based on the compatibility conditions, the displacement field of an elastomer satisfies the following relationship:

$$\nabla \times (\nabla \mathbf{u} + \mathbf{u} \nabla) \times \nabla = 0, \quad (1)$$

with \mathbf{u} the displacement and ∇ the vector operator del. According to Equation (1), the displacement field is a vector with a third derivative. However, it is not difficult to prove that the Equation (1) changes the inherent properties of vector fields. Rewriting Equation (1), the following Equation is obtained:

$$\nabla \times (\nabla \mathbf{u}) \times \nabla = -\nabla \times (\mathbf{u} \nabla) \times \nabla. \quad (2)$$

It is obtained from Equations (1) and (2) that $\nabla \times (\nabla \mathbf{u})$ and $(\mathbf{u} \nabla) \times \nabla$ can be non-zero. This is inconsistent with the properties of vector fields. For a vector field with a second derivative, like displacement field, $\nabla \times (\nabla \mathbf{u})$ and $(\mathbf{u} \nabla) \times \nabla$ are both zero.

In the classical theory of elasticity, although local rigid body rotation is acknowledged, the deformation is only taken into account for the compatibility conditions. In order to ensure the compatibility conditions of elastomer, the symmetric and antisymmetric parts of the gradient of displacement should be both considered. Since the continuous partial derivatives of multivariate functions are independent of the order of differentiation, the displacement field satisfies the following equation [17]:

$$\nabla \times (\nabla \mathbf{u}) = 0, \quad (3)$$

\mathbf{u} is a vector with a second derivative, which indicates that Equation (3) holds for an arbitrary vector with a second derivative. Therefore, Equation (3) should be the compatibility conditions followed by the theory of elasticity.

2.2 Conservation of moment of momentum by expanding stress at different point

The continuum mechanics regards the motion of material elements forming a continuum as the motion of particle. However, the introduce of the theorem of moment of momentum is actually to treat the motion of elements forming a continuum as rigid body translational motion. Here we examine the paradox arising from the introduction of conservation of moment of momentum. For classical continuum mechanics, the momentum conservation of continuum in the differential form is expressed by Cauchy's equation of motion as follows:

$$\nabla \cdot \boldsymbol{\sigma} - \boldsymbol{F} = 0, \quad (4)$$

where, $\boldsymbol{\sigma}$ is the second-order stress tensor, \boldsymbol{F} is the general body force which includes inertia force. According to the traditional interpretation of Cauchy's equation of motion, the force acting on a material element can be described as shown in Figure 1, where the stress is constant on any one of six surface elements. Equation (4) shows that a material element's size is small enough for the change in stresses on the various boundary surfaces to be approximated linearly. Because Equation (4) is independent of coordinate selection, the change of stresses on the different boundary surfaces of a material element can be described by expanding stress at any point in the material element. Figure 1a and Figure 1b show the stresses on boundary surfaces by expanding stress at the centroid of material element and at the lower left corner of material element, respectively.

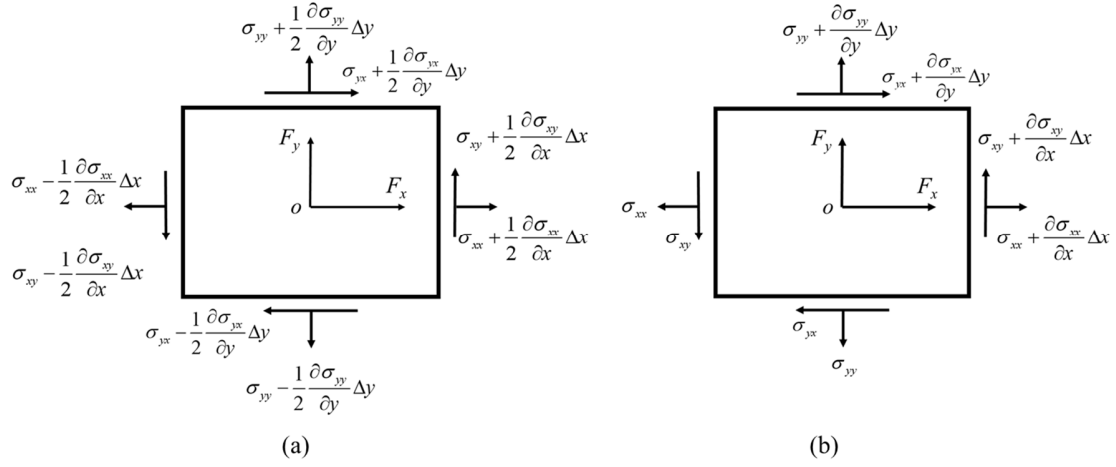


Figure 1. Diagram of force on a material element. (a) stress is expanded at the centroid of material element, (b) stress is expanded at the lower left corner of material element

When the stresses on boundary surfaces of material element are described by expanding stress at the centroid of material element, the moment of all the forces about an axis passing through the centroid of material element o and parallel to the z -axis is expressed as follows:

$$\begin{aligned}
 M_o = & \frac{1}{2} \left(\sigma_{xy} + \frac{1}{2} \frac{\partial \sigma_{xy}}{\partial x} \Delta x \right) \Delta x \Delta y \Delta z + \frac{1}{2} \left(\sigma_{xy} - \frac{1}{2} \frac{\partial \sigma_{xy}}{\partial x} \Delta x \right) \Delta x \Delta y \Delta z \\
 & - \frac{1}{2} \left(\sigma_{yx} + \frac{1}{2} \frac{\partial \sigma_{yx}}{\partial y} \Delta y \right) \Delta x \Delta y \Delta z - \frac{1}{2} \left(\sigma_{yx} - \frac{1}{2} \frac{\partial \sigma_{yx}}{\partial y} \Delta y \right) \Delta x \Delta y \Delta z .
 \end{aligned} \tag{5}$$

In Equation (5), the absence of body moments is assumed because the body force does not produce torque around the center of material element. Equation (5) can be simplified as:

$$M_o = (\sigma_{xy} - \sigma_{yx}) \Delta x \Delta y \Delta z . \tag{6}$$

Under the assumption that the material element moves as a particle, the moment of all the forces around the centroid of material element is zero. Thus,

$$\sigma_{yx} = \sigma_{xy} . \tag{7}$$

With the similar method, the equivalence of shear stress is proven, and is thought to indicate that the stress tensor is symmetric.

When the stresses on boundary surfaces of the material element are described by expanding stress at the lower left corner of material element, the moment of all forces about an axis passing through the centroid of material element o and parallel to the z -axis is written as:

$$M_o = \frac{1}{2} \left(\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial x} \Delta x \right) \Delta x \Delta y \Delta z + \frac{1}{2} \sigma_{xy} \Delta x \Delta y \Delta z - \frac{1}{2} \left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} \Delta y \right) \Delta x \Delta y \Delta z - \frac{1}{2} \sigma_{yx} \Delta x \Delta y \Delta z, \quad (8)$$

Equation (8) can be simplified as:

$$M_o = (\sigma_{xy} - \sigma_{yx}) \Delta x \Delta y \Delta z + \frac{1}{2} \left(\frac{\partial \sigma_{xy}}{\partial x} \Delta x - \frac{\partial \sigma_{yx}}{\partial y} \Delta y \right) \Delta x \Delta y \Delta z. \quad (9)$$

In the traditional derivation of the angular momentum conservation of material element with the stress expanded at the lower left corner of material element, the second term on the right side of Equation (9) is regarded as the higher order term and is dropped [1], and Equation (7) is then obtained. It should be noted that dropping the second term on right side of Equation (9) violates both the derivation of equation of motion and the limit algorithm. The equilibrium of material element along x direction is expressed as:

$$T_x = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \right) \Delta x \Delta y \Delta z, \quad (10)$$

which indicates that the resultant force acting on the material element and causing it to move is the higher-order term of stress.

In actuality, the second term on right side of Equation (9) is a higher order infinity

than the moment of inertia of the material element. The moment of material element about the axis z (I_z) is written as:

$$I_z = \rho \left((\Delta x)^2 + (\Delta y)^2 \right) \Delta x \Delta y \Delta z / 12, \quad (11)$$

with ρ the density of continuum. Therefore, in order to prevent the material element from rotating, the moment of the surface force acting on the material element relative to the centroid of the material element should be zero. Since the size of a material element is arbitrary, the following formulas hold when the moment of all forces about the selected axis is zero:

$$\sigma_{yx} = \sigma_{xy}, \quad (12)$$

$$\frac{\partial \sigma_{xy}}{\partial x} = \frac{\partial \sigma_{yx}}{\partial y}. \quad (13)$$

According to Equations (12) and (13), shear stress on one surface of a material element is equivalent to shear stress on the opposite surface, which suggests that the conservation of moment of momentum requires the stress in continuum to be constant. The constraint on stress is inconsistent with Equation (7), and is in disagreement with the stress distribution shown in Cauchy's equation of motion.

2.3 Calculation of stress on an inclined plane

Explaining the strain on an inclined plane at a point involves a similar conflict. A tetrahedron with three faces oriented in the coordinate planes and an infinitesimal plane oriented in any direction is taken into consideration in order to compute the stress on an inclined plane at a point [2, 3]. Figure 2 shows the forces on a tetrahedron with three faces oriented in the coordinate planes and with an infinitesimal plane oriented in an arbitrary

direction. The integral form of the tetrahedron's equilibrium under forces is as follows:

$$\oint_S d\mathbf{S} \cdot \mathbf{T} - \oint_V \mathbf{F} dV = 0, \quad (14)$$

here, \mathbf{T} is the stress on the surface of tetrahedron.

Since the volume of the tetrahedron is infinitesimal, the stresses on each of the tetrahedron is regarded as constants. The equilibrium of the tetrahedron under forces is rewritten as:

$$\mathbf{T}_n \Delta S = \mathbf{T}_1 \Delta S_1 + \mathbf{T}_2 \Delta S_2 + \mathbf{T}_3 \Delta S_3 + \mathbf{F} \Delta V, \quad (15)$$

where, \mathbf{T}_n , and ΔS are the stress on the inclined plane and the area of the inclined plane, \mathbf{T}_1 , \mathbf{T}_2 and \mathbf{T}_3 are the stresses on the three planes oriented in the coordinate planes, respectively, ΔS_1 , ΔS_2 and ΔS_3 are the area of three planes oriented in the coordinate planes, respectively. Neglecting the body force in the tetrahedron, the stress on an inclined plane is obtained by the equilibrium of forces acting on the tetrahedron and is written as:

$$\mathbf{T} = \mathbf{n} \cdot \boldsymbol{\sigma}, \quad (16)$$

with \mathbf{n} the unit vector of outer normal of an inclined plane.

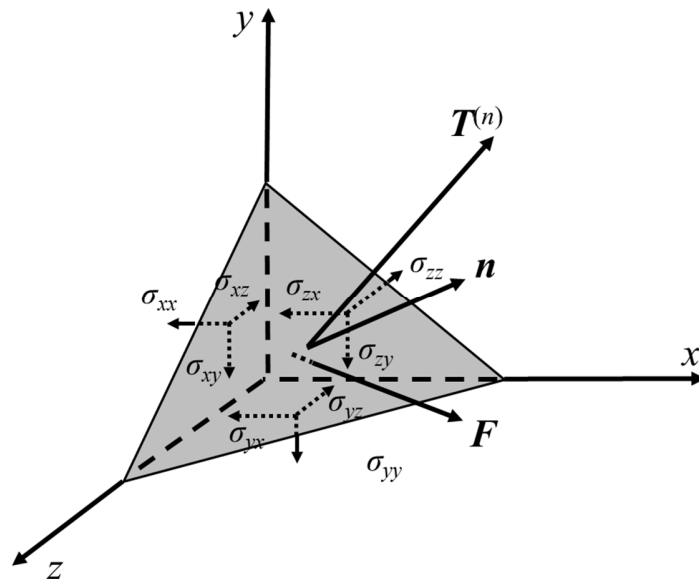


Figure 2. Equilibrium of forces on tetrahedron

As the shape of material elements is not constrained in defining the momentum conservation of material components, the equation for determining the stress on an inclined plane is the momentum conservation equation of material elements. The calculation of the stress on an inclined plane is equivalent to denying the necessity of body force in Cauchy's equation of motion and deeming no waves in elastic media. The purpose of this study is to propose a new motion description of continuum to solve these paradoxes. In classical mechanics, a rigid body is defined as a system of mass points subject to the holonomic constraints [18]. The introduction of the theorem of moment of momentum illustrates that continuum mechanics regards the material elements forming a continuum as rigid bodies that do not undergo rotation from a dynamic perspective. This conflicts with the fact that the displacement and velocity fields can be curl fields, because the antisymmetric part of the displacement gradient and velocity gradient is considered to be the rotation of the material element. The study believes that continuum mechanics should regard the material elements as deformable bodies that only undergoes centroid translation under external forces. This means that the displacement of a material element should be regarded as the displacement of the centroid of the element or the average of displacement of the element. The rotation of displacement field indicates that there is a difference in the vertical direction of the motion of material elements.

3. Differential description of Newton's second law of motion

When the momentum conservation of a discrete body is described in classical mechanics, the resultant force acting on the discrete body is given by the vector sum. Figure

3 shows a force system on a block, which is composed of two forces operating vertically on the ends of the block in the x direction. The momentum conservation of the block is expressed by the sum of vectors as:

$$\mathbf{T}_1 - \mathbf{T}_2 = m\mathbf{a} , \quad (17)$$

here, \mathbf{T}_1 and \mathbf{T}_2 are the external force acting on both ends of the block, m is the mass of block and \mathbf{a} is the average acceleration of the block. Given that the block is a rigid body, the internal force acting on its transversal section along the x direction should rise linearly as illustrated in Figure 1b. The motion of the block under an external force can be described with the motion of any free body included within the block.

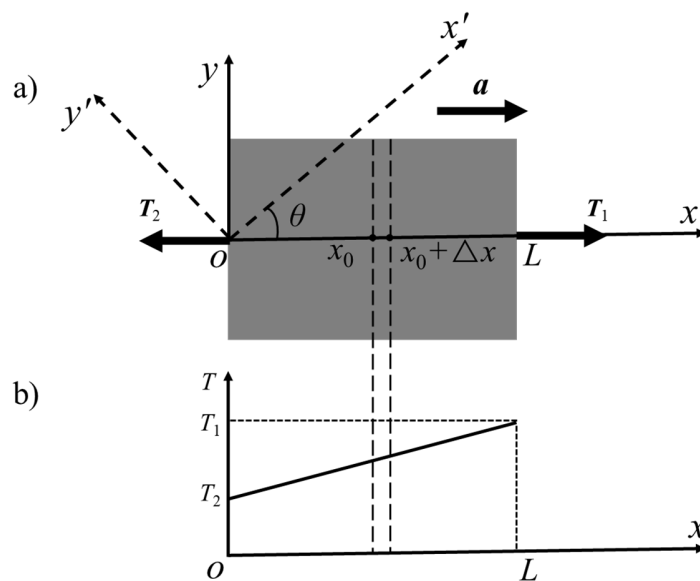


Figure 3. Force analysis of a block. (a) Forces on the block, (b) internal force on the transversal section of the block along the x direction

Assuming that there is a free body with the length Δx located at x_0 as illustrated in Figure 3, The momentum conservation of the free body can be expressed by the sum of

vectors as:

$$\frac{T_1 - T_2}{L} \Delta x = \frac{\Delta x}{L} m a , \quad (18)$$

here, L is the length of block. The momentum conservation of a free body can be expressed in differential form when its volume approaches toward zero. The momentum conservation of a free body with infinitesimal volume can be expressed in differential form directly using vector operation as:

$$\nabla \cdot T \hat{x} = \rho a , \quad (19)$$

where, T is the force on the surface of free body along the x direction, \hat{x} is the unit vector of x coordinate and ρ is the density of block. Equation (19) shows that the motion direction should be known beforehand when the momentum conservation of a free body with infinitesimal volume is given in differential form by directly taking the divergence for the internal force field. This is not feasible in continuum mechanics. In order to express the conservation of momentum of a free body with infinitesimal volume in differential form, a new operation must be introduced. This issue is addressed in an attempt by Cauchy's equation of motion. Unfortunately, the way that the divergence of second-order tensors is currently calculated is wrong., the section gives the correct expression of divergence of second-order tensor based on the assumption of continuum motion. The correctness of the new expression of divergence of second-order tensor will be verified in the next section through tensor analysis.

For the forced block in Figure 3, the momentum conservation of a free body with infinitesimal volume in the block can be written in differential form as:

$$\nabla \cdot \sigma^S = \rho a , \quad (20)$$

here, ρ is the density of block, σ^S is a second-order symmetric tensor whose components

are expressed with matrix at the Cartesian coordinate system (x, y, z) as:

$$\sigma_{ij}^S = \begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (21)$$

with T the value of normal force on the transversal section of the block along x direction. Since the motion of a block is independent of the selection of coordinates, the momentum conservation of a free body described with Equation (20) should not change with coordinate selection. At the Cartesian coordinate system (x', y', z') , the components of σ^S are expressed with matrix as:

$$\sigma_{ij}^S = \begin{bmatrix} T \cos^2 \theta & T \cos \theta \sin \theta & 0 \\ T \cos \theta \sin \theta & T \sin^2 \theta & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (22)$$

with θ the rotation angle of the coordinate system (x', y', z') relative to the coordinate system (x, y, z) . Given that Equation (20) describes motion irrespective of coordinate choice, the following equation should be true:

$$\nabla \cdot \sigma^S = \nabla \left(tr(\sigma^S) \right), \quad (23)$$

here, $tr(\sigma^S)$ is the trace of σ^S , which is the first invariant of σ^S . Equation (23) distinguishes between the divergence of a second order symmetric tensor and the divergence of a second order symmetric tensor in continuum mechanics. It is obvious that a vector would be expanded to a second-order tensor once it is realized that pressure is sometimes considered as a spherical tensor and other times as a scalar.

When the forces illustrated in Figure 3 act on a block, the internal force field in the block is a curl free field because the circulation of surface force acting on any free body inside the block is zero. Figure 4 illustrates a force system on a block, which consists of

two forces acting tangentially on the ends of the block in the x direction. In contrast to the external forces acting on the block as illustrated in Figure 3, the circulation of surface force acting on the block is non-zero. Assuming that the block is deformable and the acceleration of any free body inside the block is along the direction of resultant force, the conservation of momentum of a free body with infinitesimal volume in the block can be expressed in differential form as:

$$\nabla \times (\boldsymbol{\varepsilon} : \boldsymbol{\sigma}^A) = \rho \mathbf{a}, \quad (24)$$

here, $\boldsymbol{\varepsilon}$ is Levi Civita symbol and $\boldsymbol{\sigma}^A$ is a second-order asymmetric tensor whose components can be described by a matrix at the cartesian coordinate system (x, y, z) and at the Cartesian coordinate system (x', y', z') respectively as:

$$\sigma_{ij}^A = \begin{bmatrix} 0 & T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (25)$$

and

$$\sigma_{ij}^A = \begin{bmatrix} T \cos \theta \sin \theta & T \cos^2 \theta & 0 \\ -T \sin^2 \theta & -T \cos \theta \sin \theta & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (26)$$

It is easy to prove that $\boldsymbol{\varepsilon} : \boldsymbol{\sigma}^A$ is a vector independent of the coordinate selection.

Assuming that the motion of material elements forming a continuum is the superposition of two motions specified above, the conservation of momentum for a continuum should be expressed in differential form as follows:

$$\nabla \left(\text{tr}(\boldsymbol{\sigma}^S) \right) - \nabla \times (\boldsymbol{\varepsilon} : \boldsymbol{\sigma}^A) = \rho \mathbf{a}. \quad (27)$$

Comparing the conservation of momentum described by Equation (27) with the wave equation in the theory of elasticity and the Navier-Stokes equation in the theory of fluid

dynamics, in which the momentum conservation of elastomers and fluids are described with displacement and velocity, it is seen that they are the same in form [2, 14].

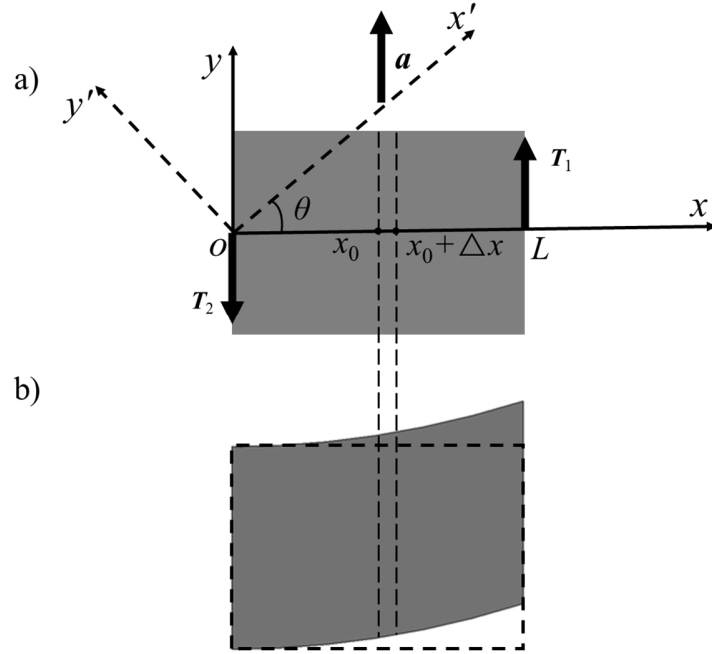


Figure 4. Force analysis of a block. (a) Forces on the block, (b) motion of the block under external force

4. Relationship between the properties of a vector field and its gradient

Assuming that A represents an arbitrary vector field, two vectors infinitesimally close in distance satisfy the following connection under linear expansion:

$$A(\mathbf{R} + \delta\mathbf{R}) = A(\mathbf{R}) + \delta\mathbf{R} \cdot \nabla A, \quad (28)$$

where, \mathbf{R} is the radius vector and $\delta\mathbf{R}$ is the increment of radius vector. Separating the gradient of A into three tensors: a spherical tensor α , a deviatoric tensor α' and a rotation tensor χ , they can be expressed with the gradient of A as:

$$\alpha = \frac{1}{6} \text{tr}(\nabla A + A\nabla) \mathbf{I}, \quad (29)$$

$$\boldsymbol{\alpha}' = \frac{1}{2}(\nabla \boldsymbol{A} + \boldsymbol{A} \nabla) - \boldsymbol{\alpha}, \quad (30)$$

$$\boldsymbol{\chi} = \frac{1}{2}(\nabla \boldsymbol{A} - \boldsymbol{A} \nabla), \quad (31)$$

where, \boldsymbol{I} represents second order unit tensor, $\boldsymbol{A} \nabla$ represents the transposition of $\nabla \boldsymbol{A}$. When vector field \boldsymbol{A} represents displacement field in the theory of elasticity, the spherical tensor $\boldsymbol{\alpha}$, deviatoric tensor $\boldsymbol{\alpha}'$ and rotation tensor $\boldsymbol{\chi}$ correspondingly describe the volume expansion, shear deformation and rigid body rotation of material element.

With Equations (29) to (31), the divergence and curl of \boldsymbol{A} can be expressed as:

$$\nabla \cdot \boldsymbol{A} = \text{tr}(\boldsymbol{\alpha} + \boldsymbol{\alpha}' + \boldsymbol{\chi}) = \text{tr}(\boldsymbol{\alpha}), \quad (32)$$

$$\nabla \times \boldsymbol{A} = \boldsymbol{\varepsilon} : (\boldsymbol{\alpha} + \boldsymbol{\alpha}' + \boldsymbol{\chi}) = \boldsymbol{\varepsilon} : \boldsymbol{\chi}. \quad (33)$$

Equations (32) and (33) shows that the divergence and curl of a vector field are included in spherical tensor and rotation tensor, respectively. When the gradient of \boldsymbol{A} is a symmetric tensor, the vector field \boldsymbol{A} is a curl free field, and when the gradient of \boldsymbol{A} is an asymmetric tensor, the vector field \boldsymbol{A} is a curl field. By taking the divergence of a gradient, the Laplace operator is obtained. The Laplacian of a vector field is another vector field and is expressed as:

$$\nabla^2 \boldsymbol{A} = \nabla \nabla \cdot \boldsymbol{A} - \nabla \times \nabla \times \boldsymbol{A}. \quad (34)$$

Submitting Equations (22) and (33) into Equation (34), the Laplacian of the vector field \boldsymbol{A} can be rewritten as:

$$\nabla^2 \boldsymbol{A} = \nabla (\text{tr}(\boldsymbol{\alpha})) - \nabla \times (\boldsymbol{\varepsilon} : \boldsymbol{\chi}). \quad (35)$$

That is, the Laplacian of a vector field can be described by the spherical tensor and rotation tensor of its gradient.

In continuum mechanics, the Laplacian of displacement is obtained by directly taking the divergence of strain (rate) tensor, a second-order symmetric tensor [14, 15]. By taking the divergence of symmetric and antisymmetric parts of the gradient of \mathbf{A} , the following formulas can be obtained:

$$\nabla \cdot (\boldsymbol{\alpha} + \boldsymbol{\alpha}') = \frac{1}{2} (\nabla^2 \mathbf{A} + \nabla \nabla \cdot \mathbf{A}) = \nabla \nabla \cdot \mathbf{A} - \frac{1}{2} \nabla \times \nabla \times \mathbf{A}, \quad (36)$$

$$\nabla \cdot \boldsymbol{\chi} = \frac{1}{2} (\nabla^2 \mathbf{A} - \nabla \nabla \cdot \mathbf{A}) = -\frac{1}{2} \nabla \times \nabla \times \mathbf{A}. \quad (37)$$

Equations (36) and (37) both contain curl part of vector \mathbf{A} . The result conflicts with the original intention of the theory of elasticity to separate the local rigid body rotation from deformation of elastomer through tensor decomposition. With Equation (36), the divergences of spherical tensor and deviatoric tensor are expressed as follows:

$$\nabla \cdot \boldsymbol{\alpha} = \nabla \nabla \cdot \mathbf{A}, \quad (38)$$

$$\nabla \cdot \boldsymbol{\alpha}' = -\frac{1}{2} \nabla \times \nabla \times \mathbf{A}. \quad (39)$$

$\boldsymbol{\alpha}'$ is a second order symmetric tensor, which should have been independent of the curl of the field \mathbf{A} . It is obtained from Equations (37) and (39) that the typical method of determining the divergence of the second-order tensor is in conflict with the traditional notion that deformation is independent of rotation. In order to distinguish between a curl field and a curl free field, the gradient of a vector field should be divided into an asymmetric tensor obtained from the gradient of a curl field and asymmetric tensor obtained from the gradient of a curl free field. In this case, the Laplacian of the decomposed vector field equals the divergence of its gradient.

5. Theories of elasticity and fluid dynamics from new dynamic hypothesis

The section re-establishes the theories of elasticity and fluid dynamics from new dynamic hypothesis. The study believes that the general method to continuum mechanics is sound, but introduces a superfluous dynamic hypothesis, conservation of moment of momentum, and makes errors in the derivation of divergence of second order tensor. The section concentrates on the rectification of the constitutive relationship of elastomer and fluid as well as the derivation of the elastic wave equation and Navier-Stokes equation by substituting the displacement and velocity into new motion equation, respectively.

5.1 New motion description

According to the analysis of the relationship between a vector field and its gradient in previous section shows, the stress tensor can be expressed with a vector. Here we assume that the stress tensor can fully capture the stress state at a point in a continuum, and that it can be written as the gradient of a vector field Σ termed as the stress vector here:

$$\sigma = \nabla \Sigma . \quad (40)$$

The characteristic of the stress vector Σ controls the circulation of the surface force acting on the free body within a continuum. When the stress vector Σ is a curl free field, the circulation of surface force acting on the free body is zero and the stress tensor is a symmetric tensor. When the stress vector Σ is a curl field, the circulation of surface force acting on the free body is non-zero and the stress tensor is an asymmetric tensor. With Equation (40), the divergence of stress tensor can be rewritten with Σ as:

$$\nabla \cdot \sigma = \nabla \nabla \cdot \Sigma - \nabla \times \nabla \times \Sigma . \quad (41)$$

In accordance of Equation (35), Equation (41) can be rewritten as:

$$\nabla \cdot \boldsymbol{\sigma} = \nabla \left(\text{tr}(\boldsymbol{\sigma}^S) \right) - \nabla \times (\boldsymbol{\varepsilon} : \boldsymbol{\tau}), \quad (42)$$

where, $\boldsymbol{\sigma}^S$ is the symmetric part of $\boldsymbol{\sigma}$, and $\boldsymbol{\tau}$ is the antisymmetric part of $\boldsymbol{\sigma}$, which are respectively expressed with $\boldsymbol{\Sigma}$ as:

$$\boldsymbol{\sigma}^S = \frac{1}{2}(\nabla \boldsymbol{\Sigma} + \boldsymbol{\Sigma} \nabla), \quad (43)$$

$$\boldsymbol{\tau} = \frac{1}{2}(\nabla \boldsymbol{\Sigma} - \boldsymbol{\Sigma} \nabla). \quad (44)$$

According to the properties of the displacement field when transverse waves propagate in elastomer and the properties of the velocity field when viscous fluid flows, the study believes that the momentum conservation described with Equation (27) can fully describe the momentum conservation of classical continuum:

$$\rho \frac{D\boldsymbol{v}}{Dt} - \nabla^2 \boldsymbol{\Sigma} - \boldsymbol{f} = 0, \quad (45)$$

here, D/Dt is the material derivative, \boldsymbol{v} is the velocity of material element translation, \boldsymbol{f} is the body force which is a curl free field. Replacing the stress vector $\boldsymbol{\Sigma}$ with stress tensor, the conservation of momentum of classical continuum is rewritten as:

$$\nabla \left(\text{tr}(\boldsymbol{\sigma}^S) \right) - \nabla \times (\boldsymbol{\varepsilon} : \boldsymbol{\tau}) + \boldsymbol{f} = \rho \frac{D\boldsymbol{v}}{Dt}. \quad (46)$$

It is clear from the new motion description of the material element that the stress tensor does not need to be symmetric in classical continuum mechanics. The stress tensor is asymmetric when the circulation of stress field is not zero.

Due to the introduction of unsuitable dynamic hypothesis in continuum mechanics, the analysis of the properties of stress field is impossible. Until now, the characteristics of the stress field have not yet been fully analyzed. The investigation of stress field properties

is made possible by the introduction of the stress vector $\boldsymbol{\Sigma}$. With Equation (40), the stress on an inclined plane symbolised with \boldsymbol{T} can be expressed as:

$$\boldsymbol{T} = \boldsymbol{n} \cdot \boldsymbol{\sigma} = \frac{\boldsymbol{\Sigma}(\boldsymbol{R} + \delta\boldsymbol{R}) - \boldsymbol{\Sigma}(\boldsymbol{R})}{|\delta\boldsymbol{R}|}. \quad (47)$$

Equation (47) shows that the stress on an inclined plane is the directional derivative of stress vector $\boldsymbol{\Sigma}$. The stress tensor is an introduced mathematical concept to conveniently describe the stress on a surface at a point and establishes the relationship between stress and deformation of continua.

5.2 Derivation of elastic wave equation from new theory of elasticity

For an elastomer with small deformation, the convective acceleration is zero, and the conservation of momentum can be simplified as:

$$\nabla \left(\text{tr}(\boldsymbol{\sigma}^S) \right) - \nabla \times (\boldsymbol{\varepsilon} : \boldsymbol{\sigma}^A) + \boldsymbol{f} = \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2}, \quad (48)$$

here, $\partial/\partial t$ is the time derivative and ρ is a constant. Under the new motion description, the study believes that the constitutive relation and strain-displacement relationship of isotropic elastomer should be expressed as follows:

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{e}, \quad (49)$$

$$\boldsymbol{e} = \nabla \boldsymbol{u}, \quad (50)$$

here, \boldsymbol{C} is the fourth-order elastic tensor and \boldsymbol{e} is the strain tensor including the traditionally defined one and the rotation tensor. The elastic tensor in component form should be expressed as:

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl}. \quad (51)$$

It is seen from Equations (49) to (51) that the stress tensor is asymmetric when

displacement field is rotational and is symmetric when displacement field is curl free. In fact, that no stress produced by local rigid body rotation makes the traditional representation of the elastic tensor superfluous. Though the traditional elastic tensor is replaced with Equation (51), the stress tensor is symmetric only if strain tensor is symmetric.

By submitting Equation (49) into Equation (48), the momentum conservation of an elastomer is expressed with strain tensor as:

$$(\lambda + \mu)\nabla(\text{tr}(\mathbf{e}^s)) - \mu\nabla \times (\boldsymbol{\varepsilon} : \boldsymbol{\Omega}) + \mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \quad (52)$$

with

$$\mathbf{e}^s = \frac{1}{2}(\nabla \mathbf{u} + \mathbf{u} \nabla), \quad (53)$$

$$\boldsymbol{\Omega} = \frac{1}{2}(\nabla \mathbf{u} - \mathbf{u} \nabla), \quad (54)$$

where, \mathbf{e}^s and $\boldsymbol{\Omega}$ are the strain tensor and rotation tensor. Via the relationship between vector field and its gradient (Equations (32) and (33)), Equation (52) can be rewritten with displacement as:

$$(\lambda + \mu)\nabla \nabla \cdot \mathbf{u} - \mu \nabla \times \nabla \times \mathbf{u} + \mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}. \quad (55)$$

It is seen from Equation (55) that the wave equation derived from new conservation of momentum can predict both the existence of longitudinal wave and transverse wave in elastomer, while the velocity of longitudinal wave is lower than that of traditional one.

5.3 Derivation of Navier-Stokes equation from new theory of fluid dynamics

For viscous fluids, the stress is related to the volume deformation and shear motion.

When the shear motion occurs in a viscous fluid, the velocity field is a curl field. In this case, the internal force field in the viscous fluid is a curl field, and the stress tensor is an asymmetric tensor. When the stress brought on by shear motion is distinguished from the stress brought on by volume deformation, the momentum conservation of fluid is expressed as follows:

$$\nabla(\text{tr}(\boldsymbol{\sigma})) - \nabla \times (\boldsymbol{\varepsilon} : \boldsymbol{d}) + \boldsymbol{f} = \rho \frac{D\boldsymbol{v}}{Dt}, \quad (56)$$

with

$$\boldsymbol{d} = \boldsymbol{\sigma} - \frac{1}{3} \text{tr}(\boldsymbol{\sigma}) \boldsymbol{I}, \quad (57)$$

here, \boldsymbol{d} is deviatoric stress tensor which is an asymmetric tensor. For a Newtonian fluid, the relation between deviatoric stress tensor and deviatoric strain rate tensor $\boldsymbol{\xi}$ in component form is expressed as:

$$d_{ij} = \eta \delta_{ik} \delta_{jl} \xi_{kl}, \quad (58)$$

where, η is the viscosity of fluid. The deviatoric strain rate tensor $\boldsymbol{\xi}$ and the velocity \boldsymbol{v} have the following relationship:

$$\boldsymbol{\xi} = \nabla \boldsymbol{v} - \frac{1}{3} \text{tr}(\nabla \boldsymbol{v}) \boldsymbol{I}. \quad (59)$$

Submitting Equation (59) into Equation (58) and consequently submitting the relation between deviatoric stress and deviatoric strain rate expressed with velocity into Equation (56), the momentum conservation of a fluid can be expressed with velocity as:

$$\nabla(\text{tr}(\boldsymbol{\sigma})) - \eta \nabla \times (\boldsymbol{\varepsilon} : \nabla \boldsymbol{v}) + \boldsymbol{f} = \rho \frac{D\boldsymbol{v}}{Dt}. \quad (60)$$

Since the following relations hold:

$$\nabla \times (\boldsymbol{\varepsilon} : \nabla \boldsymbol{v}) = \nabla \times \nabla \times \boldsymbol{v} = -\nabla^2 \boldsymbol{v} + \nabla \nabla \cdot \boldsymbol{v}, \quad (61)$$

Equation (60) can be rewritten as:

$$\nabla(\text{tr}(\boldsymbol{\sigma})) + \eta(\nabla^2 \mathbf{v} - \nabla \nabla \cdot \mathbf{v}) + \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt}. \quad (62)$$

In accordance with the definition of pressure, the following equation holds:

$$p = -\frac{1}{3} \text{tr}(\boldsymbol{\sigma}), \quad (63)$$

with p defined pressure. Equation (62) is rewritten as:

$$-3\nabla p + \eta(\nabla^2 \mathbf{v} - \nabla \nabla \cdot \mathbf{v}) + \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt}. \quad (64)$$

Equation (64), while different, has the same in form as the Navier-Stokes equation for Newtonian fluid. The difference between Equation (64) and Navier-Stokes equation indicates that there is a difference between thinking about pressure as a scalar and a spherical tensor. When pressure is regarded as a scalar symbolled P , the relationship between fluid volume deformation and pressure is expressed as:

$$P = -K \nabla \cdot \mathbf{u}, \quad (65)$$

With K the bulk modulus of fluid. When pressure is regarded as a spherical tensor, the relationship between fluid volume deformation and pressure is expressed as:

$$p\mathbf{I} = -\frac{K}{3}(\nabla \cdot \mathbf{u})\mathbf{I}. \quad (66)$$

It is seen from Equations (65) and (66) that $P=3p$. The pressure in Navier-Stokes equation is the pressure in scalar form P . Submitting Equation (66) into Equation (64) and replacing velocity with displacement, Equation (64) is rewritten as follows:

$$K \nabla \nabla \cdot \mathbf{u} + \eta(\nabla^2 \dot{\mathbf{u}} - \nabla \nabla \cdot \dot{\mathbf{u}}) + \mathbf{f} = \rho \frac{D\dot{\mathbf{u}}}{Dt}. \quad (67)$$

with $\dot{\mathbf{u}} = \partial \mathbf{u} / \partial t$. For further special case of incompressible fluid, the mass conservation

reduces to $\nabla \cdot \mathbf{v} = 0$, which means that the density of fluid does not change with pressure, Equation (64) reduces to:

$$-\nabla P + \eta \nabla^2 \mathbf{v} + \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt}. \quad (68)$$

Equation (65) is the equation of motion for incompressible Newtonian fluid, which is the same with Navier-Stokes equation.

From Equation (60), it is seen that the viscous force is generated only by the shear motion of fluid (or relative slide of fluid elements) rather than traditional defined shear deformation of fluid. As a result, Newton's definition of the viscous force differs significantly from the one derived from that based on traditional deformation theory. There are two key distinctions between the viscous forces as defined by Newton and the viscous forces as defined by traditional deformation theory. One is that the former demonstrates that a scalar potential cannot adequately represent the fluid flow's velocity field, whereas the latter demonstrates that a scalar potential may adequately characterize the flow's potential to produce viscous force. Another difference is that the former believes that viscous force is related to local rigid body rotation, while the latter believes that viscous force is independent of local rigid body rotation. Batchelor appears to have discovered the distinction. He pointed out that there is a paradox in the description of viscous force with deformation theory that viscous force should be independent of the local vorticity [2]. In the Navier-Stokes equation of motion, the viscous force is only related to the curl velocity field. This illustrates that the viscous force defined by Newton is appropriate rather than that defined by deformation theory.

When the fluid flow is described with the theory of fluid dynamics based on traditional deformation theory, the bulk viscosity is added in the viscous coefficient tensor, which

indicates that the viscous force can be caused by bulk deformation of fluid [2, 16]. In order to make the thermodynamic pressure equal to the mechanical pressure in fluid, the Stokes hypothesis is introduced. In fact, during the formulation of Navier-Stokes equation of motion, the stress caused by volume deformation (or pressure) has been distinguished from stress tensor. The term describing viscous force in the Navier-Stokes equation of motion should, therefore, not include viscous force related to volume deformation. This again demonstrates that the theory of traditional fluid dynamics has problems in explaining the fluid dynamics.

6. Application of new theory of elasticity

We take the problem of a wedge subjected to dead-weight and liquid pressure as an example to verify the validity of the new theory of elasticity. As shown in Figure 5, the height of the wedge is L , the top angle of the wedge is α and the unit volume weight of the wedge is ρg (ρ is the density of wedge). the liquid pressure on the wedge at depth y is equal to γy (γ is the density of the liquid, g is the gravitational acceleration). In traditional theory of elasticity, the analytical expression of assumed stress function is often obtained by introducing boundary conditions of the edges of $o-a$ and $o-b$ [19]. The boundary condition of the edge of $a-b$ is often not considered. If the problem is regarded as the problem of a variable cross-section beam under distributed load, it can be found that the solution does not consider the influence of reaction force on the stress distribution in the beam. The same problem also occurs in solving cantilever beam problems. Here We solve the stress distribution of the wedge by adding constraints on the edge of $a-b$. To obtain an analytical solution to the problem, the additional boundary conditions that the vertical displacement at point a and the horizontal displacement at point b are zero are added.

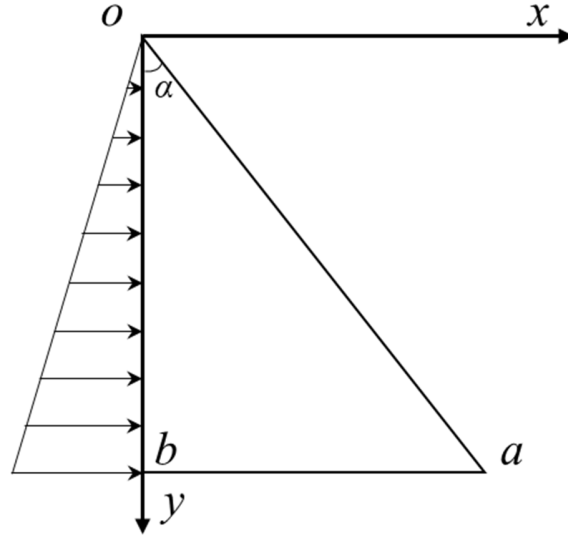


Figure 5. diagram of a wedge subjected to dead-weight and liquid pressure.

Assuming that the stress potential in the plane (xoy) can be assumed as follows:

$$\Sigma_x = \frac{\partial \phi}{\partial x} + \frac{\partial \varphi}{\partial y}, \quad (69)$$

$$\Sigma_y = \frac{\partial \phi}{\partial y} - \frac{\partial \varphi}{\partial x}, \quad (70)$$

here ϕ and φ are the scalar potential and vector potential, the two potentials can be expressed as follows, respectively:

$$\phi = Ax^3 + Bx^2y + Cxy^2 + Dy^3, \quad (71)$$

$$\varphi = Px^3 + Qx^2y + Mxy^2 + Ny^3. \quad (72)$$

here, A, B, C, D, P, Q, M, N are the coefficient determined by boundary conditions.

Assuming that the thickness of the wedge is very large, the problem can be regarded as a plane strain problem. With Equations (49) and (50), the displacement in the plane (xoy) can be expressed as:

$$u_x = \frac{\partial \chi}{\partial x} + \frac{1}{\mu} \frac{\partial \varphi}{\partial y}, \quad (73)$$

$$u_y = \frac{\partial \chi}{\partial y} - \frac{1}{\mu} \frac{\partial \varphi}{\partial x}, \quad (74)$$

with χ a scalar potential, which can be assumed as:

$$\chi = Ex^3 + \frac{1}{\mu} (Bx^2y + Cxy^2) + Fy^3, \quad (75)$$

here, E and F are the coefficient determined by the constitutive relation of an elastomer.

According to Equations (40) (49) and (50), the following equations are obtained:

$$F = \mu D = -\frac{1}{3\mu} B, \quad (76)$$

$$E = -\frac{1}{\mu} A = -\frac{1}{3\mu} C. \quad (77)$$

Replacing D and A in Equations (71) and (72) with B and C , the components of stress tensor are expressed as:

$$\left\{ \begin{array}{l} \sigma_{xx} = 2Cx + 2By + 2Qx + 2My \\ \sigma_{xy} = 2Bx + 2Cy - 6Px - 2Qy \\ \sigma_{yx} = 2Bx + 2Cy + 2Mx + 6Ny \\ \sigma_{yy} = 2Cx - \frac{2}{\mu^2} By - 2Qx - 2My - \rho gy \end{array} \right. , \quad (78)$$

Replacing E and F in Equation (75) with B and C , the components of displacement are expressed as:

$$\left\{ \begin{array}{l} u_x = \frac{2Bxy + C(y^2 - x^2)}{\mu} + \frac{Qx^2 + 2Mxy + 3Ny^2}{\mu} \\ u_y = \frac{B(x^2 - y^2) + 2Cxy}{\mu} - \frac{3Px^2 + 2Qxy + My^2}{\mu} \end{array} \right. , \quad (79)$$

Submitting the boundary conditions of the wedge, the coefficient are obtained. According to the boundary conditions of the edge of $o-b$ ($\sigma_{xx}=-\gamma gy$, $\sigma_{xy}=0$ when $x=0$), the following equations are obtained:

$$B + M = -\frac{\gamma g}{2}, \quad (80)$$

$$C - Q = 0. \quad (81)$$

According to the boundary conditions of the edge of $o-a$ ($\mathbf{n} \cdot \boldsymbol{\sigma} = 0$ when $x=y \tan \alpha$), the following equations are obtained:

$$\sin \alpha C + \left(\cos \alpha - \frac{\sin^2 \alpha}{\cos \alpha} \right) B + \left(1 - \frac{\sin^2 \alpha}{\cos \alpha} \right) M - 3 \sin \alpha N = 0, \quad (82)$$

$$\left(1 + \frac{1}{\mu^2} \right) B - 3P + M = -\frac{\rho g}{2}. \quad (83)$$

According to the boundary conditions at the point b ($u_x=0$), the following equation is obtained:

$$C + 3N = 0. \quad (84)$$

According to the boundary conditions at the point a ($u_y=0$), the following equation is obtained:

$$\left(\tan^2 \alpha - 1 \right) B + 2 \tan \alpha C - 3 \tan^2 \alpha P - 2 \tan \alpha Q - M = 0, \quad (85)$$

Rewriting Equations (80)-(85) in matrix form, the linear equation system composed of undetermined coefficients is expressed as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ \cos \alpha - \frac{\sin^2 \alpha}{\cos \alpha} & \sin \alpha & 0 & 0 & 1 - \frac{\sin^2 \alpha}{\cos \alpha} & -3 \sin(\alpha) \\ \left(1 + \frac{1}{\mu^2}\right) & 0 & -3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 3 \\ \tan^2 \alpha - 1 & 0 & -3 \tan^2 \alpha & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} B \\ C \\ P \\ Q \\ M \\ N \end{bmatrix} = \begin{bmatrix} -\frac{\gamma g}{2} \\ 0 \\ 0 \\ -\frac{\rho g}{2} \\ 0 \\ 0 \end{bmatrix}. \quad (86)$$

The undetermined coefficients are obtained by multiplying both sides of Equation (85) by the inverse of the coefficient matrix.

7. Conclusions

The study has proven that the theorem of moment of momentum is not necessary in continuum mechanics via the derivation of conservation of moment of momentum with the stress expanded at different points and that the deformation coordination equation cannot accurately reflect the properties of the displacement field by examining the equilibrium equation expressed in terms of displacement. The theories of elasticity and fluid dynamics are re-established on the basis of new dynamic hypotheses. In the new theories, the theorem of momentum is the only dynamic law that a continuum obeys. The second law of Newton, which explains the dynamic behavior of discrete particles, is extend to differential form. by examining the connection between of a vector's characteristics and its gradient, the validity of the new differential form of Newton's second law is demonstrated. The result shows that the stress tensor can be written as an asymmetric tensor and with a gradient of a vector. the new theories of elasticity and fluid dynamics are established by correctly altering the constitutive relationship of elastomer and fluid, respectively. The compatibility conditions and Stokes hypothesis no longer required in the new theories of elasticity and

fluid dynamics, respectively. It is shown that traditional deformation theory is not appropriate for the describing viscous force and that Newton's definition of viscous force differs significantly from that based on classic deformation theory. A curl force field, like viscous force, can balance a curl free force field, like pressure, in a continuum according the Navier-Stokes equation. The problem of a wedge subjected to dead-weight and liquid pressure is also solved with the new theory of elasticity to verify the validity of the new theory of elasticity.

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Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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