

## Revisiting fundamental equations of fluid flow

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**Abstract.** The study rederives the fundamental equations of fluid flow and examines the inherent relationship between momentum conservation and mechanical energy conservation. It is shown that the material derivative of velocity is to depict the acceleration of fluid particles in Eulerian perspective, and momentum conservation and mechanical energy conservation are the same concept termed in different descriptions. According to the study, the traditional formula for energy conservation of fluid fails to distinguish the difference between physical quantities described in Lagrangian and Eulerian perspectives. The study simplifies the energy conservation of incompressible Newtonian fluid to be internal energy conservation.

**Keywords:** momentum conservation; energy conservation; Navier-Stokes equation; Lagrangian flow field; Eulerian flow field

## 1. Introduction

The study on viscous fluid flow is an ancient topic that may date back to prehistoric times when humans began to use streamlined tools like spears and arrows. The attempts to get the exact solution to the problem of the viscous fluid has been made since the Greek philosopher Aristotle (384-322 B.C.) and mathematician Archimedes (287-212 B.C.) [1, 2]. In 1752, Euler first used the partial differential equation to formulate the flow of fluid with negligible viscosity [1, 3]. Subsequently, by adopting Newton's definition of friction, Navier and Stokes independently include the viscous forces into the fluid dynamic equations for inviscid fluids, and the fluid dynamic equations for viscous fluids are obtained, which are now called Navier-Stokes equations now. Since Navier-Stokes equations can well describe some special flows, such as laminar pipe flows and some boundary layer flows [4, 5], Navier-Stokes equations are considered the fundamental equations of fluid dynamics. In 2000, seeking the general solution to Navier-Stokes equations was selected as one of seven Millennium Problems by the Clay Mathematics Institute of Cambridge, U.S. (<http://www.claymath.org/millennium-problems>). In 2008, the U.S. Defense Advanced Research Projects Agency (DARPA) listed it as one of 23 DARPA Mathematical Challenges— “Mathematical Challenge Four: 21st Century Fluids”.

Fluid mechanics is believed to be a branch of classical mechanics. In fluid dynamics, the motion of fluid elements is regarded as the motion of particles, which can be described with Newton's three laws of motion or other mechanical principles

related to and equivalent to them [2, 5]. In fluid mechanics, the motion of a fluid is described in two different manners: the Eulerian description and the Lagrangian description. Navier-Stokes equations are the equations of motion for fluids, in which the viscous force is expressed with velocity. In Navier-Stokes equations, pressure, velocity, acceleration, etc. are quantities in the Eulerian description. In the description, the momentum of a fluid element is considered a function with respect to time and location coordinates [5]. The acceleration of a fluid element in Eulerian perspective is a combination of local acceleration corresponding to the change of momentum at a point with time and convective acceleration corresponding to the change of momentum at a point with location coordinates.

In accordance with Newton's laws of motion, a fluid remains stationary or in a state of uniform linear motion when no external force acts on the fluid, and the motion state of the fluid changes when external forces act on it. The flow of a fluid under the influence of boundaries is the velocity change of fluid particles under boundary resistance. When Navier-Stokes equations describe the steady flow of an ideal fluid, the equation of mechanical energy conservation, Bernoulli's principle, is derived [5]. Since Bernoulli's principle, the conservation law of mechanical energy for ideal fluids, is obtained from the momentum conservation of ideal fluids, there should be an inherent correlation between momentum conservation and energy conservation, and the inherent correlation should be related to the manner in which the motion of an object is described.

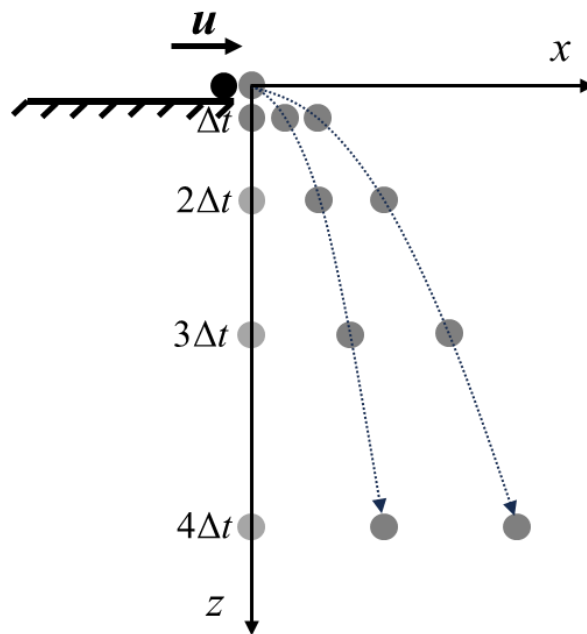
In classical mechanics, momentum is the product of a system's mass and its velocity [6]. It is a vector and has the same direction as velocity. Based on Newton's

second law of motion, the change in momentum of a system equals the external resultant force multiplied by the time over which its momentum changes. Kinetic energy, a form of energy, is the product of a system's mass and the square of its velocity [6]. It is a scalar. Based on the kinetic energy theorem, the change in kinetic energy equals the external resultant force multiplied by the displacement over which its momentum changes. This indicates that force is described under different perspectives in momentum conservation and energy conservation: force is regarded as a function with respect to time in momentum conservation and a function with respect to position in energy conservation. If there is an underlying connection between relationship between momentum conservation and mechanical energy conservation, the classical fluid energy conservation equation may be broken down into the conservation of mechanical energy and the conservation of internal energy. The possible decomposition of energy conservation for fluid may simplify the numerical calculation of heat transfer based on fluid transport.

The study rederives the fundamental equations of fluid flow by examining the innate connection between mechanical energy conservation and momentum conservation. In the study, Both the acceleration and the momentum conservation of a mass are expressed from an Eulerian perspective. It is revealed that the momentum conservation is equivalent to the conservation of mechanical energy. Then, the energy conservation of fluid is derived by expressing quantities in Eulerian perspective, and the energy conservation is simplified to be the internal energy conservation.

## 2. Relationship between velocities from Lagrangian and Eulerian perspectives

Considering the motion of free falling that an object with horizontal velocity is falling under the sole influence of gravity, the trajectory and landing point of the object are determined by its initial velocity as shown in Fig. 1. For a motion of free falling with a known initial velocity, the spatial position and time of the object satisfy a one-to-one correspondence during the falling process. Therefore, for any motion of free falling, the velocity and acceleration of the object can be described as a function of its spatial position during its falling process. Under the concept of modern physics, which regards fields as the media of forces, quantities in mechanics like velocity and force are described from the Eulerian perspective as functions with respect to location coordinates.



**Fig. 1 Diagrams of free fall motion**

In Lagrangian perspective, the velocities of a particle at two infinitely close

moments satisfy the following relationship:

$$\mathbf{u}(t + \delta t) = \mathbf{u}(t) + \frac{d\mathbf{u}}{dt} \delta t \quad (1)$$

here,  $\mathbf{u}$  is the velocity vector in Lagrangian perspective,  $t$  is the time,  $\delta t$  is the time increment, and  $d/dt$  is the time derivative. In Eulerian perspective, the velocities of a particle at two points that are infinitely close satisfy the following relationship:

$$\mathbf{U}(\mathbf{x} + \delta \mathbf{x}) = \mathbf{U}(\mathbf{x}) + \delta \mathbf{x} \cdot \nabla \mathbf{U} \quad (2)$$

here,  $\mathbf{U}$  is the velocity vector in Eulerian perspective which is a function with respect to location coordinates,  $\mathbf{x}$  is the position vector,  $\delta \mathbf{x}$  is the displacement of a particle during  $\delta t$ ,  $\nabla$  is the gradient operator.

Since the following relationship holds at a point:

$$\mathbf{U} = \mathbf{u} = \frac{\delta \mathbf{x}}{\delta t} \quad (3)$$

The velocities described in Lagrangian perspective and in Eulerian perspective meet the following relationship:

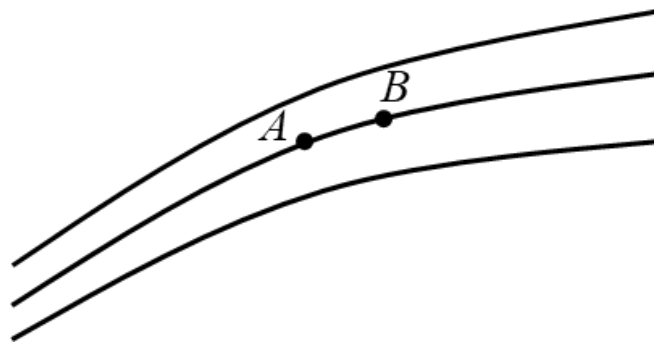
$$\frac{d\mathbf{u}}{dt} = \mathbf{U} \cdot \nabla \mathbf{U} \quad (4)$$

in view of the vector identity [2, 5]

$$\mathbf{U} \cdot \nabla \mathbf{U} = \frac{1}{2} \nabla (U^2) + \nabla \times \mathbf{U} \times \mathbf{U} \quad (5)$$

with  $U$  the modulus of  $\mathbf{U}$ , Equation (4) is rewritten as:

$$\frac{d\mathbf{u}}{dt} = \frac{1}{2} \nabla (U^2) + \nabla \times \mathbf{U} \times \mathbf{U} \quad (6)$$



### **Fig. 2 Diagram of streamlines in a steady flow**

For a steady flow, the streamline coincides with the trace, and the velocity at a point in Eulerian perspective is equal to the velocity of a fluid particle at the time that the particle is located at the point in Lagrangian perspective. Hence, the velocity at a streamline can be described as the velocity of a particle moving along the streamline at different times. Fig. 2 shows a diagram of streamlines in a steady flow. *A* and *B* represent two points on a streamline. In accordance with Equation (4), the convective acceleration is the acceleration of a fluid particle described from the Eulerian perspective, by which the acceleration of a fluid particle at a certain moment is obtained from the spatial distribution of velocity in the flow field. Therefore, Navier-Stokes equations are the equation of motion for fluid in Eulerian perspective, where quantities like velocity, pressure and body force are all field quantities.

### **3. Correlation between momentum conservation and conservation of mechanical energy**

In classical mechanics, the following law holds:

$$\mathbf{f} \delta t = m \delta \mathbf{u} \quad (7)$$

here,  $\mathbf{f}$  is the resultant force on an object,  $m$  is the mass of the object which is a constant. Equation (7) is termed the impulse-momentum theorem, by which Newton's second law is obtained. We know that the external forces described in Lagrangian perspective at a moment are equal to the external forces described in Eulerian perspective at the position where the object is located at this moment. According to the theorem of kinetic



energy, the following equation holds:

$$\mathbf{F} \cdot \delta \mathbf{x} = m \mathbf{U} \cdot \nabla \mathbf{U} \cdot \delta \mathbf{x} \quad (8)$$

with  $\mathbf{F}$  the resultant force in Eulerian perspective. Since  $\mathbf{U}$  is parallel to  $\delta \mathbf{x}$ , which can be obtained from Equation (3), The following equation is obtained:

$$\mathbf{U} \times (\nabla \times \mathbf{U}) \cdot \delta \mathbf{x} = 0 \quad (9)$$

Then, Equation (8) is rewritten as follows:

$$\mathbf{F} \cdot \delta \mathbf{x} = \frac{1}{2} m \nabla (U^2) \cdot \delta \mathbf{x} \quad (10)$$

Equation (10) is the conservation of mechanical energy. Since Equation (10) is obtained from Equation (7) (the equation of momentum conservation), momentum conservation should be equivalent to the conservation of mechanical energy.

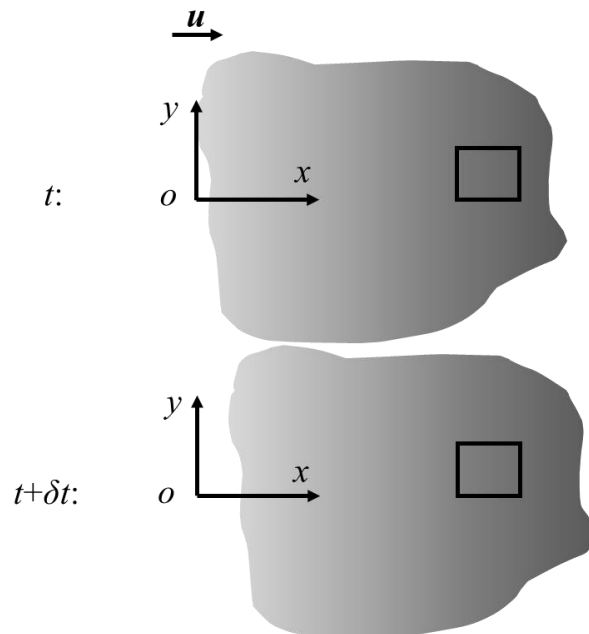
According to Newton's second law, the direction of the inertial acceleration of a mass should coincide with the direction of the consequent force exerted upon it. The gradient of a scalar field is a vector field and whose magnitude is the rate of change and which points in the direction of the greatest rate of increase of the scalar field. When the dynamics of a particle are described in field perspective, that is, describing the dynamics of a particle under Eulerian perspective, the gradient direction of kinetic energy is not the direction of the resultant force acting on a particle. Taking Poiseuille flow as example, the gradient direction of kinetic energy field is perpendicular to the streamline of the flow field. Therefore, the second term on the right side of Equation (6) is the velocity change related to the velocity difference between adjacent streamlines at a point, which has nothing to do with the inertial acceleration of fluid particles. This

means that the first term on the right side of Equation (6) contains the parts unrelated to particle acceleration, and the second term on the right side of Equation (6) is meant to balance off the parts in the first term.

When the force field is a time-varying field, the acceleration of a particle passing through a specific point in space at different times is different. In this case, the velocities described in Lagrangian perspective and in Eulerian perspective meet the following relationship [2]:

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \quad (11)$$

which is called the material derivative of velocity. Equation (11) makes clear that, from an Eulerian perspective, the acceleration of fluid particle is made up of two parts: the velocity field's fluctuation over time and the variation of velocity field in space.



**Fig. 3 Diagram of a moving fluid with a changing density**

Fig.3 shows a moving fluid with a changing density. In Fig. 3, the fluid density

distribution is represented by the distribution of grayscale. Fig. 3 shows that regardless of whether the fluid is compressible or not, the mass within a tiny volume in space changes continuously as the fluid travels. Then, the mass conversation in Eulerian perspective is obtained:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0 \quad (12)$$

with  $\rho$  is the density of a fluid at a point. When the density of a fluid is uniform and does not change under forces, Equation (12) is simplified as:

$$\nabla \cdot \mathbf{U} = 0 \quad (13)$$

In this case, mass conversation requires the volume of fluid particle to remain constant.

When the fluid moves uniformly, Equation (12) is simplified as:

$$\frac{\partial \rho}{\partial t} + \mathbf{U} \cdot \nabla \rho = 0 \quad (14)$$

In this case, the density distribution of fluid is what causes the change in density at a given location. A particle's momentum is determined by multiplying its mass by its velocity, hence the change in momentum at a given location should be related with the change in density at that location.

In Lagrangian perspective, the rate of momentum change of fluid particles is expressed as follows:

$$\frac{d(\bar{\rho} v \mathbf{u})}{dt} = \frac{d(\bar{\rho} v)}{dt} \mathbf{u} + \bar{\rho} v \frac{d\mathbf{u}}{dt} \quad (15)$$

here,  $\bar{\rho}$  is the density of the fluid particle and  $v$  is the volume of the fluid particle.

Since the mass conversation of a fluid particle in Lagrangian perspective is expressed as:

$$\frac{d(\bar{\rho}v)}{dt} = 0 \quad (16)$$

Equation (15) is simplified as:

$$\frac{d(\bar{\rho}v\mathbf{u})}{dt} = \bar{\rho}v \frac{d\mathbf{u}}{dt} \quad (17)$$

With the material derivative, the rate of momentum change of a fluid particle with a volume of one unit can be stated in Eulerian perspective as:

$$\frac{d(\bar{\rho}\mathbf{u})}{dt} = \rho \frac{\partial \mathbf{U}}{\partial t} + \rho \mathbf{U} \cdot \nabla \mathbf{U} \quad (18)$$

By decomposing the resultant force acting on a fluid particle with a volume of one unit into pressure, viscous force and body force, the Navier-Stokes equation of motion for Newtonian fluid is obtained [5]:

$$\rho \frac{\partial \mathbf{U}}{\partial t} + \rho \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla P + \mu \nabla^2 \mathbf{U} + \rho \mathbf{G} \quad (19)$$

here,  $P$  is the pressure,  $\mu$  is the viscous coefficient and  $\mathbf{G}$  is the body force on per volume.

For incompressible Newtonian fluids, the following vector identity holds:

$$\nabla^2 \mathbf{U} = -\nabla \times \nabla \times \mathbf{U} \quad (20)$$

Submitting Equations (5) and (20) into Equations (19), Equations (19) is rewritten as:

$$\rho \frac{\partial \mathbf{U}}{\partial t} + \rho \frac{1}{2} \nabla (U^2) + \rho \nabla \times \mathbf{U} \times \mathbf{U} = -\nabla P - \mu \nabla \times \nabla \times \mathbf{U} + \rho \mathbf{G} \quad (21)$$

Equation (19) shows that the viscous force only depends on the current flow field and is unaffected by temporal variations in the flow field. This suggests that at a specific moment in time when the velocity field distribution is known, the fluctuation of the velocity field over time affects the distribution of the pressure gradient.

#### 4. Internal energy conservation of a fluid particle in Eulerian perspective

The first law of thermodynamics states that the change in a system's internal energy is equal to the difference between the heat added to the system from its surroundings and the work done on the system. That is:

$$dE = dQ + dW \quad (22)$$

where,  $E$  is the total energy,  $Q$  and  $W$  are the added heat and the work done from its surroundings, respectively. For fluid, the total energy of the system is a flowing fluid particle and  $E$  will include internal energy and kinetic energy. Thus, for a fluid particle, the rate of time change of energy per unit volume is expressed in Eulerian perspective as:

$$\frac{dE}{dt} = \rho \left( \frac{\partial e}{\partial t} + \mathbf{U} \cdot \left( \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right) \right) \quad (23)$$

with  $e$  the internal energy per unit mass, which is a field quantity related to time and spatial coordinates.

The net rate of work done on a fluid particle is expressed in Eulerian perspective as:

$$\frac{dW}{dt} = (-\nabla P + \mu \nabla^2 \mathbf{U} + \rho \mathbf{G}) \cdot \mathbf{U} \quad (24)$$

Equation (24) is different from the traditional expression of the net rate of work done on a fluid particle located at a certain point. This means that the traditional expression failed to distinguish the quantities in Lagrangian perspective from the quantities in Eulerian perspective. It should be pointed out that the kinetic energy of fluid particles converts into internal energy when viscous force does work on a fluid particle.

Therefore, the work done by viscous forces does not change the total energy of fluid.

In Eulerian perspective, the heat-transfer term is expressed as [2]:

$$\frac{dQ}{dt} = \nabla(k\nabla T) - \mu\nabla^2\mathbf{U} \cdot \mathbf{U} \quad (25)$$

where,  $k$  is the thermal conductivity and  $T$  is the temperature. The second term on the right side of Equation (25) is the internal heat generated from energy conversion.

Submitting Equations (24) - (25) into Equation (22), Equation (22) is written as:

$$\mathbf{U} \cdot \left( \rho \frac{\partial \mathbf{U}}{\partial t} + \rho \mathbf{U} \cdot \nabla \mathbf{U} + \nabla P - \mu \nabla^2 \mathbf{U} - \rho \mathbf{G} \right) + \rho \frac{\partial e}{\partial t} - \nabla(k\nabla T) + \mu \nabla^2 \mathbf{U} \cdot \mathbf{U} = 0 \quad (26)$$

With Equation (12), Equation (20) is simplified as:

$$\rho \frac{\partial e}{\partial t} - \nabla(k\nabla T) + \mu \nabla^2 \mathbf{U} \cdot \mathbf{U} = 0 \quad (27)$$

Equation (27) is the internal energy conservation of a fluid particle in Eulerian perspective. It is seen from Equation (27) that only the work done on fluid by the viscous force is converted into internal energy of the fluid and pressure does not cause changes in internal energy when doing work on a fluid. This is because the effect of internal energy changes on pressure have not been considered in the mechanical energy conservation equation of the fluid, Equation (19).

## 5. Conclusions

In summary, the study rederived the fundamental equations of fluid flow by investigating the intrinsic relationship between momentum conservation and mechanical energy conservation. It is discovered that momentum conservation and mechanical energy conservation are two different terms for the same concept, and the material derivative of velocity field is to describe the acceleration of fluid particles in

Eulerian perspective. The gradient direction of kinetic energy is not the direction of the resultant force applied on a particle when the dynamics of a particle are described from an Eulerian perspective. The study rederived energy conservation of incompressible fluid by expressing mechanical energy conservation in Eulerian perspective. The energy conservation of incompressible fluid is simplified as the internal energy conservation. It is shown that the traditional approach to energy conservation failed to distinguish the quantities in Lagrangian perspective from the quantities in Eulerian perspective. Currently, physics often uses fields to describe material interactions. The conclusion in the study has a driving effect on understanding the various fields in physics.

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### **Conflict of interest**

The authors have no conflicts to disclose.

### **Data availability**

There is no new data in the study.

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