A SHORT-COURSE ON

Nonlinear Continuum Mechanics

J. Tinsley Oden

CAM 397
Introduction to Mathematical Modeling
Third Edition
Fall 2008

Preface

These notes are prepared for a class on an Introduction to Mathematical Modeling, designed to introduce CAM students to Area C of the CAM-CES Program. I view nonlinear continuum mechanics as a vital tool for mathematical modeling of many physical events - particularly for developing phenomenological models of thermomechanical behavior of solids and fluids. I attempt here to present an accelerated course on continuum mechanics accessible to students equipped with some knowledge of linear algebra, matrix theory, and vector calculus. I supply notes and exercises on these subjects as background material.

J. Tinsley Oden
September 2008

CONTENTS

Chapter 1

Kinematics of Deformable Bodies

Continuum mechanics models the physical universe as a collection of "deformable bodies," a concept that is easily accepted from our everyday experiences with observable phenomena. Deformable bodies occupy regions in three-dimensional Euclidean space $\mathcal{E}$, and a given body will occupy different regions at different times. The subsets of $\mathcal{E}$ occupied by a body $\mathcal{B}$ are called its *configurations*. It is always convenient to identify one configuration in which the geometry and physical state of the body are known and to use that as the *reference configuration*; then other configurations of the body can be characterized by comparing them with the reference configuration (in ways we will make precise later).

For a given body, we will assume that the reference configuration is an open, bounded, connected subset Ω_0 of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega_0$. The body is made up of physical points called *material points*. To identify these points, we assign each a vector $\mathbf{X}$ and we identify the components of $\mathbf{X}$ as the coordinates of the place occupied by the material point when the body is in its reference configuration relative to a fixed Cartesian coordinate system.

It is thus important to understand that the body $\mathcal{B}$ is a *non-denumerable* set of material points $\mathbf{X}$. This is the fundamental hypothesis of continuum mechanics: matter is not discrete, it is continuously distributed in one-to-one correspondence with points in some subset of $\mathbb{R}^3$. Bodies are thus "continuous media" – the components of $\mathbf{X}$ with respect to some basis are real numbers. Symbolically, we could write

$$\mathcal{B} = \{\mathbf{X}\} \sim \{\mathbf{X} = X_i\mathbf{e}_i \ \in \ \bar{\Omega}_0\}$$

for some orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and origin $\mathbf{0}$ chosen in three-dimensional Euclidean space and, thus, identified with $\mathbb{R}^3$. Hereafter, repeated indices are summed throughout their ranges: the "summation convention".

Kinematics is the study of the motion of bodies, without regard to the causes of the motion. It is purely a study of geometry and is an exact science within the hypothesis of a *continuum* (a continuous media).

1.1 Motion

We pick a point $\mathbf{0}$ in $\mathbb{R}^3$ as the origin of a fixed coordinate system $(x_1, x_2, x_3) = \mathbf{x}$ defined by orthonormal vectors $\mathbf{e}_i$, $i = 1, 2, 3$. The system (x_1, x_2, x_3) is called the *spatial* coordinate system. When the physical body $\mathcal{B}$ occupies its reference configuration Ω_0 at, say, time $t = 0$, the material point $\mathbf{X}$ occupies a position (place) corresponding to the vector $\mathbf{X} = X_i\mathbf{e}_i$.

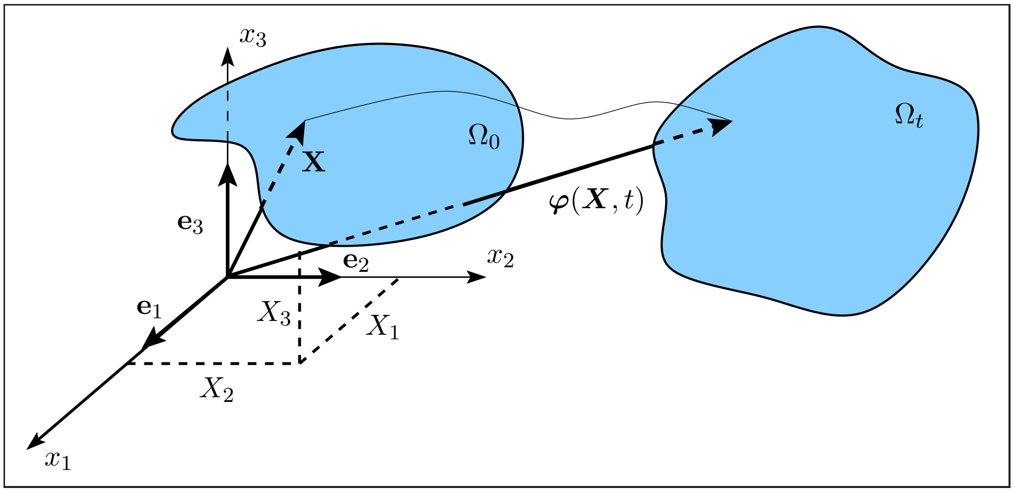

Figure 1.1: Motion from the reference configuration Ω_0 to the current configuration Ω_t.

The spatial coordinates (X_1, X_2, X_3) of $\mathbf{X}$ are *labels* that identify the material point. The coordinate labels X_i are sometimes called material coordinates (see Fig. 1.1).

Remark: Notice that if there were a *countable* set of discrete material points, such as one might use in models of molecular or atomistic dynamics, the particles (discrete masses) could be labeled using natural numbers $n \in \mathbb{N}$, as indicated in Figure 1.2. But the particles (material points) in a continuum are not countable, so the use of a label of three real numbers for each particle corresponding to the coordinates of their position (at $t = 0$) in the reference configuration, seems to be a very natural way to identify such particles.

The body moves through $\mathcal{E}$ over a period of time and occupies a configuration $\Omega_t \subset \mathbb{R}^3$ at time t. Thus, material points $\mathbf{X}$ in $\bar{\Omega}_0$ (the closure of Ω_0) are mapped into positions $\mathbf{x}$ in $\bar{\Omega}_t$ by a smooth vector-valued mapping (see Fig. 1.1)

$$\boxed{\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t)}$$

Thus, $\boldsymbol{\varphi}(\mathbf{X}, t)$ is the spatial position of the material point $\mathbf{X}$ at time t. The one-parameter family $\{\boldsymbol{\varphi}(\mathbf{X}, t)\}$ of positions is called the trajectory of $\mathbf{X}$. We demand that $\boldsymbol{\varphi}$ be differentiable, injective, and orientation preserving. Then $\boldsymbol{\varphi}$ is called the *motion* of the body:

1. Ω_t is called the current configuration of the body.

2. $\boldsymbol{\varphi}$ is injective (except possibly at the boundary $\partial\Omega_0$ of Ω_0).

3. $\boldsymbol{\varphi}$ is *orientation preserving* (which means that the physical material cannot penetrate itself or reverse the orientation of material coordinates – which means that $\det \nabla\boldsymbol{\varphi}(\mathbf{X}, t) > 0$).

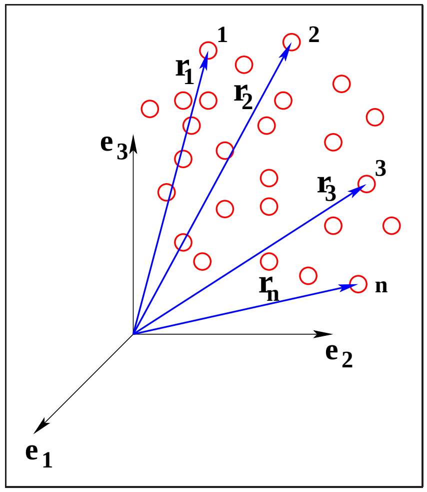

Figure 1.2: A discrete set of material particles

Hereafter we will not explicitly show the dependence of φ and other quantities on time t unless needed; this time dependency is taken up later.

The vector field

$$\boxed{\mathbf{u} = \varphi(\mathbf{X}) - \mathbf{X}}$$

is the *displacement* of point $\mathbf{X}$. Note that

$$d\mathbf{x} = \nabla\varphi(\mathbf{X})d\mathbf{X} \qquad \left(i.e.\ \ dx_i = \frac{\partial \varphi_i}{\partial X^j} dX_j\right).$$

The tensor

$$\boxed{\mathbf{F}(\mathbf{X}) = \nabla\varphi(\mathbf{X})}$$

is called the *deformation gradient*. Clearly,

$$\boxed{\mathbf{F}(\mathbf{X}) = \mathbf{I} + \nabla\mathbf{u}(\mathbf{X})}$$

where $\mathbf{I}$ is the identity tensor and $\nabla\mathbf{u}$ is the *displacement gradient*.

Some Definitions:

- A *deformation* is homogeneous if $\mathbf{F} = \mathbf{C} =$ constant

- A motion is *rigid* if it is the sum of a translation $\mathbf{a}$ and a rotation $\mathbf{Q}$:

$$\boldsymbol{\varphi}(\mathbf{X}) = \mathbf{a} + \mathbf{Q}\mathbf{X},$$

  where $\mathbf{a} \in \mathbb{R}^3$, $\mathbf{Q} \in \mathbb{O}^3_+$, with $\mathbb{O}^3_+$ the set of orthogonal matrices of order 3 with determinant equal to $+1$.

- As noted earlier, the fact that the motion is orientation preserving means that

$$\det \nabla\boldsymbol{\varphi}(\mathbf{X}) > 0 \qquad \forall\ \mathbf{X} \in \bar{\Omega}_o$$

- Recall that

$$\text{Cof } \mathbf{F} \;=\; \text{cofactor matrix (tensor) of } \mathbf{F} \;=\; \det \mathbf{F}\ \mathbf{F}^{-T}$$

  For any matrix $\mathbf{A} = [A_{ij}]$ of order n, and for each row i and column j, let $\mathbf{A}'_{ij}$ be the matrix of order $n-1$ obtained by deleting the ith row and jth column of $\mathbf{A}$. Let $d_{ij} = (-1)^{i+j}\ \det \mathbf{A}'_{ij}$. Then the matrix

$$\text{Cof}\,\mathbf{A} = [d_{ij}]$$

  is the cofactor matrix of $\mathbf{A}$ and d_{ij} is the (i,j)–cofactor of $\mathbf{A}$.

$$\boxed{\mathbf{A}(\text{Cof }\mathbf{A})^T = (\text{Cof }\mathbf{A})^T\mathbf{A} = (\det \mathbf{A})\mathbf{I}}$$

1.2 Strain and Deformation Tensors

A differential material line segment in the reference configuration is

$$dS_0^2 = d\mathbf{X}^T d\mathbf{X} = dX_1^2 + dX_2^2 + dX_3^2$$

while the same material line in the current configuration is

$$dS^2 \;=\; d\mathbf{x}^T d\mathbf{x} \;=\; d\mathbf{X}^T\mathbf{F}^T\mathbf{F}d\mathbf{X}$$

The tensor

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \text{the } \textit{right Cauchy-Green deformation tensor}$$

is thus a measure of the change in dS_0^2 due to (gradients of) the motion

$$dS^2 - dS_0^2 = d\mathbf{X}^T\mathbf{C}d\mathbf{X} - d\mathbf{X}^T d\mathbf{X}.$$

$\mathbf{C}$ is symmetric, positive definite. Another deformation measure is simply

$$dS^2 - dS_0^2 = d\mathbf{X}^T(2\mathbf{E})d\mathbf{X}$$

where

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \text{the } \textit{Green strain tensor}$$

Since $\mathbf{F} = \mathbf{I} + \nabla\mathbf{u}$ and $\mathbf{C} = \mathbf{F}^T\mathbf{F}$,

$$\boxed{\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T + \nabla\mathbf{u}^T\nabla\mathbf{u})}$$

The tensor

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \text{the } \textit{left Cauchy-Green deformation tensor}$$

is also symmetric and positive definite.

1.2.1 Interpretation of E

Take $dS_0 = dX_1$ (i.e. $d\mathbf{X} = (dX_1, 0, 0)^T$). Then

$$dS^2 - dS_0^2 = dS^2 - dX_1^2 = 2E_{11}dX_1^2,$$

so

$$E_{11} = \frac{1}{2}\left(\left(\frac{ds}{dX_1}\right)^2 - 1\right) = \begin{cases} \text{a measure of the stretch of a} \\ \text{material line originally oriented} \\ \text{in the } X_1\text{–direction in } \Omega_0 \end{cases}$$

We call e_1 the extension in the X_1–direction at $\mathbf{X}$ (which is a dimensionless measure of change-in-length-per-unit length)

$$e_1 \overset{\text{def}}{=} \frac{dS - dX_1}{dX_1} = \sqrt{1 + 2E_{11}} - 1$$

or

$$2E_{11} = (1 + e_1)^2 - 1$$

Similar definitions apply to E_{22} and E_{33}.

Now take $d\mathbf{X} = (dX_1, dX_2, 0)^T$ and

$$\cos\theta = \frac{d\mathbf{x}_1 \cdot d\mathbf{x}_2}{\|d\mathbf{x}_1\|\,\|d\mathbf{x}_2\|} = \frac{C_{12}}{\sqrt{1 + 2E_{11}}\sqrt{1 + 2E_{22}}} \qquad \text{(Exercise)}$$

The *shear* (or *shear strain*) in the X_1–X_2 plane is defined by the angle change (see Figure 1.3),

$$\gamma_{12} \overset{\text{def}}{=} \frac{\pi}{2} - \theta$$

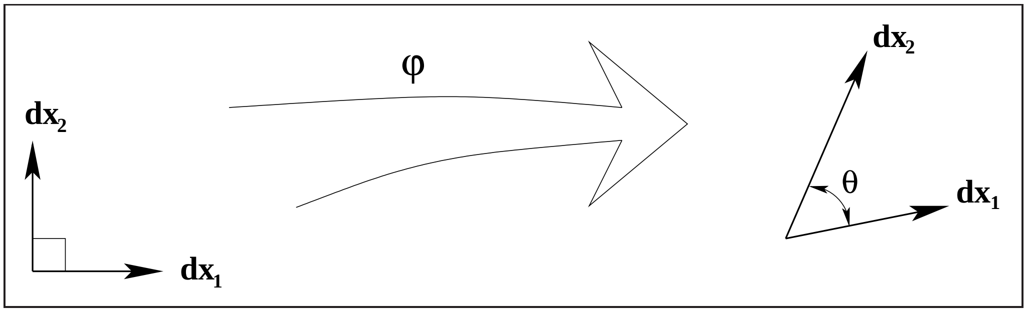

Figure 1.3: Change of angle through the motion φ.

Therefore

$$\boxed{\sin\gamma_{12} = \frac{2E_{12}}{\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}}}$$

Thus, E_{12} (and, analogously, E_{13} and E_{23}) is a measure of the shear in the X_1–X_2 (or X_1–X_3 and X_2–X_3) plane.

1.2.2 Small Strains

The tensor

$$\mathbf{e} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$$

is called the *infinitesimal* or *small* or *engineering strain tensor*. Clearly

$$\mathbf{E} = \mathbf{e} + \frac{1}{2}\nabla\mathbf{u}^T\nabla\mathbf{u}$$

Note that if $\mathbf{E}$ is "small" (i.e. $|E_{ij}| \ll 1$).

$$\begin{aligned} e_1 &= (1+2E_{11})^{1/2} - 1 \\ &= 1 + E_{11} - 1 + 0(E_{11}^2) \\ &\approx E_{11} = e_{11} \end{aligned}$$

that is

$$e_{11} = e_1 = \frac{dS - dX_1}{dX_1}, \quad \text{etc.}$$

and

$$2e_{12} = \sin\gamma_{12} \approx \gamma_{12}, \quad \text{etc.}$$

Thus, small strains can be given the classical textbook interpretation: e_n is the change in length per unit length and e_{12} is the change in the right angle between material lines in the X_1– and X_2–directions.

1.3 Rates of Motion and Deformation

If $\boldsymbol{\varphi}(\mathbf{X}, t)$ is the motion (of $\mathbf{X}$ at time t), i.e.

$$\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t)$$

then

$$\dot{\mathbf{x}} \stackrel{\text{def}}{=} \frac{\partial \boldsymbol{\varphi}(\mathbf{X}, t)}{\partial t}$$

is the velocity and

$$\ddot{\mathbf{x}} \stackrel{\text{def}}{=} \frac{\partial^2 \boldsymbol{\varphi}(\mathbf{X}, t)}{\partial t^2}$$

is the acceleration.

Since $\boldsymbol{\varphi}$ is (in general) bijective, we can also describe the velocity as a function of the place $\mathbf{x}$ in $\mathbb{R}^3$ and time t:

$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t) = \dot{\mathbf{x}}(\boldsymbol{\varphi}^{-1}(\mathbf{x}, t), t)$$

This is called the *spatial description* of the velocity.

This leads to two different ways to interpret the rates-of-motion of continua:

1. The Material Description (functions are defined on *material points* $\mathbf{X}$ in $\mathbb{R}^3$).
2. The Spatial Description (functions are defined on (spatial) *places* $\mathbf{x}$ in $\mathbb{R}^3$).

When the equations of continuum mechanics are written in terms of the material description, the collective equations are commonly referred to as the *Lagrangian* form (formulation) of the equations (see Fig. 1.4). When the spatial description is used, the term *Eulerian* form (formulation) is used (see Fig. 1.5).

There are differences in the way rates of change appear in the Lagrangian and Eulerian formulations.

In the Lagrangian case:

$$\frac{d\boldsymbol{\varphi}(\mathbf{X}, t)}{dt} = \frac{\partial \boldsymbol{\varphi}(\mathbf{X}, t)}{\partial t} + \frac{\partial \boldsymbol{\varphi}(\mathbf{X}, t)}{\partial \mathbf{X}} \cdot \frac{\partial \mathbf{X}}{\partial t},$$

but $\dfrac{\partial \mathbf{X}}{\partial t} = 0$ because $\mathbf{X}$ is simply a label of a material point. Thus,

$$\boxed{\frac{d\boldsymbol{\varphi}(\mathbf{X}, t)}{dt} = \frac{\partial \boldsymbol{\varphi}(\mathbf{X}, t)}{\partial t}}$$

In the Eulerian case: Given a field $\boldsymbol{\psi} = \boldsymbol{\psi}(\boldsymbol{x}, t)$,

$$\frac{d\boldsymbol{\psi}(\boldsymbol{x}, t)}{dt} = \left.\frac{\partial \boldsymbol{\psi}(\boldsymbol{x}, t)}{\partial t}\right|_{\boldsymbol{x} \text{ fixed}} + \frac{\partial \boldsymbol{\psi}(\boldsymbol{x}, t)}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial t}$$

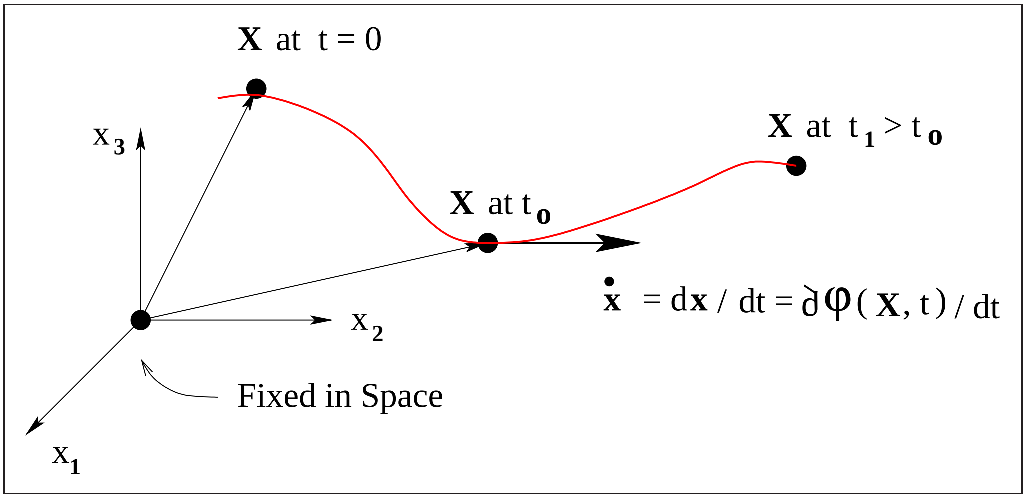

Figure 1.4: **Lagrangian (material) description of velocity:** The velocity of a material point is the time rate of change of the position of the point as it moves along its path (its trajectory) in $\mathbb{R}^3$.

but $\dfrac{\partial \boldsymbol{x}}{\partial t} = \mathbf{v}(\boldsymbol{x}, t)$ is the velocity at position $\boldsymbol{x}$ and time t. Thus,

$$\boxed{\frac{d\psi(\boldsymbol{x},t)}{dt} = \frac{\partial\psi(\boldsymbol{x},t)}{\partial t} + \mathbf{v}(\boldsymbol{x},t)\cdot\frac{\partial\psi(\boldsymbol{x},t)}{\partial \boldsymbol{x}}}$$

Lagrangian (notation)

$$\frac{\text{“}\partial\text{ ”}}{\partial \mathbf{X}} = \nabla = \text{Grad}, \qquad \frac{\partial}{\partial \mathbf{X}}\cdot\boldsymbol{\varphi} = \nabla\cdot\boldsymbol{\varphi} = \text{Div }\boldsymbol{\varphi}$$

Eulerian (notation)

$$\frac{\text{“}\partial\text{ ”}}{\partial \boldsymbol{x}} = \text{grad}, \qquad \frac{\partial}{\partial \boldsymbol{x}}\cdot\mathbf{v} = \text{div }\mathbf{v}$$

In classical literature, some authors write

$$\boxed{\frac{D\psi}{Dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}\cdot\text{grad }\psi}$$

as the “material time derivative” of a scalar field ψ, giving the rate of change of ψ at a fixed place $\mathbf{x}$ at time t. Thus, in the Eulerian formulation, the acceleration is

$$\mathbf{a} = \frac{D\mathbf{v}}{Dt} = \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\text{grad }\mathbf{v}$$

$\mathbf{v}$ being the velocity.

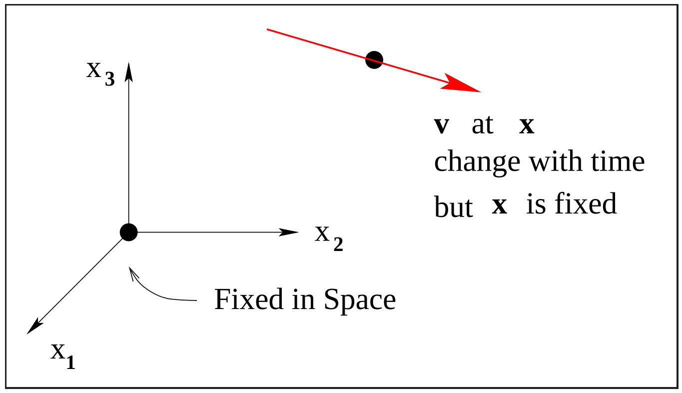

Figure 1.5: **Eulerian (spatial) description of velocity:** The velocity at a fixed place $\mathbf{x}$ in $\mathbb{R}^3$ is the speed and direction (at time t) of particles flowing through the place $\mathbf{x}$

1.4 Rates of Deformation

The spatial (Eulerian) field

$$\boxed{\mathbf{L} = \mathbf{L}(\boldsymbol{x}, t) \stackrel{\text{def}}{=} \frac{\partial}{\partial \boldsymbol{x}} \cdot \mathbf{v}(\boldsymbol{x}, t) = \text{grad } \mathbf{v}(\boldsymbol{x}, t)}$$

is the *velocity gradient*. The time rate of change of the deformation gradient $\mathbf{F}$ is

$$\begin{aligned} \dot{\mathbf{F}} &\equiv \frac{\partial}{\partial t} \nabla \boldsymbol{\varphi}(\mathbf{X}, t) = \nabla \frac{\partial \boldsymbol{\varphi}}{\partial t}(\mathbf{X}, t) \\ &= \frac{\partial}{\partial \mathbf{X}} \mathbf{v}(\boldsymbol{x}, t) = \frac{\partial \mathbf{v}}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial \mathbf{X}} = \text{grad } \mathbf{v} \ \mathbf{F} \end{aligned}$$

or,

$$\boxed{\dot{\mathbf{F}} = \text{grad } \mathbf{v} \ \mathbf{F} = \mathbf{L}_m \mathbf{F}}$$

where $\mathbf{L}_m = \mathbf{L}$ is written in material coordinates, so

$$\mathbf{L}_m = \dot{\mathbf{F}} \mathbf{F}^{-1}$$

It is standard practice to write $\mathbf{L}$ in terms of its symmetric and skew-symmetric parts:

$$\boxed{\mathbf{L} = \mathbf{D} + \mathbf{W}}$$

Here

$$\begin{aligned} \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) &= \text{the } \textit{deformation rate tensor} \\ \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) &= \text{the } \textit{spin tensor} \end{aligned}$$

We can easily show that if $\mathbf{v}$ is the velocity field,

$$\mathbf{W}\ \mathbf{v} = \frac{1}{2}\,\boldsymbol{\omega} \times \mathbf{v}$$

where $\boldsymbol{\omega}$ is the vorticity

$$\boldsymbol{\omega} = \text{curl}\ \mathbf{v}$$

Recall (Cf. Exercise 2.6) that

$$D(\det \mathbf{A}) : \mathbf{V} = (\det \mathbf{A})\mathbf{V}^T : \mathbf{A}^{-1}$$

for any invertible tensor $\mathbf{A}$ and arbitrary $\mathbf{V} \subset L(V, V)$. Also, if $\mathbf{f}(\mathbf{g}(t)) = \mathbf{f} \circ \mathbf{g}(t)$ denotes the composition of functions $\mathbf{f}$ and $\mathbf{g}$, the chain rule of differentiation leads to

$$\frac{d\mathbf{f}(\mathbf{g}(t))}{dt} = d\mathbf{f}(\mathbf{g}(t)) \cdot \frac{d\mathbf{g}(t)}{dt} = D\mathbf{f}(\mathbf{g}(t)) : \dot{\mathbf{g}}(t)$$

Combining these expressions, we have

$$\dot{\overline{\det \mathbf{F}}} = \frac{\partial \det \mathbf{F}}{\partial t} = D(\det \mathbf{F}) : \dot{\mathbf{F}} = \det \mathbf{F}\ \dot{\mathbf{F}}^T : \mathbf{F}^{-1} = \det \mathbf{F}\ \text{tr}\ \mathbf{L} = \det \mathbf{F}\ \text{div}\ \mathbf{v}$$

(Since $\dot{\mathbf{F}}^T : \mathbf{F}^{-1} = \text{tr}\ \dot{\mathbf{F}}\mathbf{F}^{-1} = \text{tr}\ \mathbf{L}_m$, where $\mathbf{L}_m$ is $\mathbf{L}$ written as a function of the material coordinates, and $\text{tr}\ \mathbf{L} = \text{tr grad}\ \mathbf{v} = \text{div}\ \mathbf{v}$). Summing up:

$$\boxed{\dot{\overline{\det \mathbf{F}}} = \det \mathbf{F}\ \text{div}\ \mathbf{v}}$$

1.5 The Piola Transformation

The situation is this: a subdomain $G_0 \subset \Omega_0$ of the reference configuration of a body, with boundary ∂G_0 and unit exterior vector $\mathbf{n}_0$ normal to the surface-area element dA_0, is mapped by the motion $\boldsymbol{\varphi}$ into a subdomain $G = \boldsymbol{\varphi}(G_0) \subset \Omega_t$ of the current configuration with boundary ∂G with unit exterior vector $\mathbf{n}$ normal to the "deformed" surface area dA (see Fig. 1.6).

Moreover, there is a tensor field $\mathbf{T}_0 = \mathbf{T}_0(\mathbf{X})$ defined on G_0 that associates with $\mathbf{n}_0$ the vector $\mathbf{T}_0(\mathbf{X})\mathbf{n}_0(\mathbf{X})$ at a point $\mathbf{X}$ on ∂G_0, the vector $\mathbf{T}_0\mathbf{n}_0$ being the *flux* of $\mathbf{T}_0$ across or through ∂G_0 at $\mathbf{X}$ with respect to $\mathbf{T}_0$. There is a corresponding tensor field $\mathbf{T}(\boldsymbol{x}) = \mathbf{T}(\boldsymbol{\varphi}(\mathbf{X}))$ defined on G that associates with $\mathbf{n}$ the vector $\mathbf{T}(\boldsymbol{x})\mathbf{n}(\boldsymbol{x})$ at a point $\boldsymbol{x} = \boldsymbol{\varphi}(\mathbf{X})$ on ∂G. *We seek a relationship between $\boldsymbol{T}_0(\boldsymbol{X})$ and $\mathbf{T}(\boldsymbol{x})$ that will result in the same total flux through the surfaces ∂G_0 and ∂G, so that*

$$\boxed{\int_{\partial G_0} \mathbf{T}_0(\mathbf{X})\mathbf{n}_0(\mathbf{X})\, dA_0 = \int_{\partial G} \mathbf{T}(\boldsymbol{x})\mathbf{n}(\boldsymbol{x})\, dA}$$

with $\boldsymbol{x} = \boldsymbol{\varphi}(\mathbf{X})$. This relationship between $\mathbf{T}_0$ and $\mathbf{T}$ is called the *Piola transformation.*

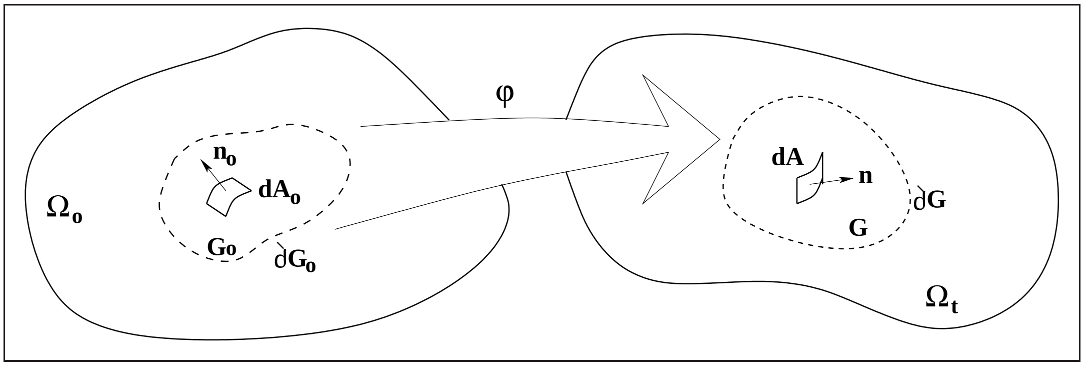

Figure 1.6: Mapping from reference configuration into current configuration.

Proposition: The above correspondence holds if

$$\boxed{\mathbf{T}_0(\mathbf{X}) = \det \mathbf{F}(\mathbf{X})\ \mathbf{T}(\boldsymbol{x})\ \mathbf{F}(\mathbf{X})^{-T} = \mathbf{T}(\boldsymbol{x})\ \mathrm{Cof}\ \mathbf{F}(\mathbf{X})}$$

Proof: (This development follows that of Ciarlet). We will use the Green's formulas (divergence theorems)

$$\int_{G_0} \mathrm{Div}\ \mathbf{T}_0\, dX = \int_{\partial G_0} \mathbf{T}_0 \mathbf{n}_0\, dA_0$$

and

$$\int_{G} \mathrm{div}\ \mathbf{T}\, dx = \int_{\partial G} \mathbf{T}\mathbf{n}\, dA$$

where

$$\mathrm{Div}\ \mathbf{T}_0 = \nabla \cdot \mathbf{T}_0 = \frac{\partial (\mathbf{T}_0)_{ij}}{\partial X_j}\mathbf{e}_i$$

$$\mathrm{div}\ \mathbf{T} = \frac{\partial}{\partial x_j} T_{ij}\mathbf{e}_i$$

$$dx = dx_1 dx_2 dx_3 = \det \mathbf{F}\ dX = \det \mathbf{F}\ dX_1 dX_2 dX_3$$

We will also need to use the fact that

$$\frac{\partial}{\partial X_j}(\mathrm{Cof}\ \nabla \boldsymbol{\varphi})_{ij} = 0$$

To show this, we first verify by direct calculation that

$$(\mathrm{Cof}\ \mathbf{F})_{ij} = (\mathrm{Cof}\ \nabla \boldsymbol{\varphi})_{ij} = \left(\frac{\partial}{\partial X_{j+i}}\varphi_{i+1}\right)\left(\frac{\partial}{\partial X_{j+2}}\varphi_{i+2}\right) - \left(\frac{\partial}{\partial X_{j+2}}\varphi_{i+1}\right)\left(\frac{\partial}{\partial X_{j+1}}\varphi_{i+2}\right)$$

where no summation is used. Then a direct computation shows that

$$\frac{\partial}{\partial X_j}(\mathrm{Cof}\ \mathbf{F})_{ij} = 0.$$

Next, set

$$\mathbf{T}_0(\mathbf{X}) = \mathbf{T}(\boldsymbol{x})\mathrm{Cof}\ \mathbf{F}(\mathbf{X})$$

Noting that

$$\frac{1}{\det \mathbf{F}}(\mathrm{Cof}\ \mathbf{F})^T = \mathbf{F}^{-1}$$

and

$$\frac{\partial x_i}{\partial X_m} \cdot \frac{\partial X_m}{\partial x_j} = \frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

we see that

$$(\mathrm{Cof}\ \mathbf{F})_{ij} = \det \mathbf{F}\ (\mathbf{F}^{-1})_{ji} = \det \mathbf{F}\ \frac{\partial X_j}{\partial x_i}$$

Thus,

$$\begin{aligned}
\mathrm{Div}\ \mathbf{T}_0(\mathbf{X}) &= \mathbf{e}_k \frac{\partial}{\partial X_k}(\mathbf{T}_0)_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \frac{\partial (\mathbf{T}_0)_{ij}}{\partial X_j}\mathbf{e}_i \\
&= \frac{\partial}{\partial X_j}((\mathbf{T}(\boldsymbol{x})\ \mathrm{Cof}\ \mathbf{F}(\mathbf{X}))_{ij}\mathbf{e}_i \\
&= \frac{\partial T_{im}(\boldsymbol{x})}{\partial x_r} \cdot \frac{\partial x_r}{\partial X_j} \cdot\ \mathrm{Cof}\ \mathbf{F}(\mathbf{X})_{mj}\mathbf{e}_i + \mathbf{T}_{im} \underbrace{\frac{\partial}{\partial X_j}\mathrm{Cof}\ \mathbf{F}(\mathbf{X})_{mj}}_{0}\ \mathbf{e}_i \\
&= \frac{\partial T_{im}}{\partial x_r} \cdot \frac{\partial x_r}{\partial X_j} \det \mathbf{F}\ \frac{\partial X_j}{\partial x_m}\mathbf{e}_i \\
&= \frac{\partial T_{ir}}{\partial x_r}\mathbf{e}_i \det \mathbf{F} \\
&= \mathrm{div}\ \mathbf{T}\ \ \det \mathbf{F}
\end{aligned}$$

that is

$$\boxed{\mathrm{Div}\ \mathbf{T}_0 = \det \mathbf{F}\ \mathrm{div}\ \mathbf{T}}$$

Thus

$$\begin{aligned}
&\int_{G_0} \mathrm{Div}\ \mathbf{T}_0\, dX = \int_{G_0} \det \mathbf{F}\ \mathrm{div}\ \mathbf{T}\, dX \\
&\int_{\partial G_0} \mathbf{T}_0 \mathbf{n}_0\, dA_0 = \int_{G_0} \mathrm{div}\ \mathbf{T} \det \mathbf{F}\, dX = \int_G \mathrm{div}\ \mathbf{T}\, dx = \int_{\partial G} \mathbf{T}\mathbf{n}\, dA
\end{aligned}$$

as asserted. □

1.6 Corollaries and Observations

- Since G_0 is arbitrary (symbolically)

$$\boxed{\mathbf{T}_0\mathbf{n}_0 \, dA_0 = \mathbf{T}\mathbf{n} \, dA}$$

- Set $\mathbf{T} = \mathbf{I} =$ identity. Then

$$\boxed{\det \mathbf{F} \; \mathbf{F}^{-T}\mathbf{n}_0 \, dA_0 = \mathbf{n} \, dA}$$

- Since $\mathbf{n} = \dfrac{dA_0}{dA} \cdot (\det \mathbf{F})\mathbf{F}^{-T}\mathbf{n}_0$ and $\|\mathbf{n}\| = 1$,

$$\boxed{dA = \det \mathbf{F} \; \|\mathbf{F}^{-T}\mathbf{n}_0\| dA_0} \qquad \text{(Nanson's Formula)}$$

where $\| \cdot \|$ denotes the Eulerian norm. Thus

$$\boxed{n = \frac{\text{Cof } \mathbf{F}\mathbf{n}_0}{\|\text{Cof } \mathbf{F}\mathbf{n}_0\|}}$$

1.7 The Polar Decomposition Theorem

A real invertible matrix $\mathbf{F}$ *can be factored in a unique way as*

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$$

where $\mathbf{R}$ *is an orthogonal matrix and* $\mathbf{U}$ and $\mathbf{V}$ *are symmetric positive definite matrices.*

Proof: (We will use as a fact the following lemma: for every symmetric positive definite matrix $\mathbf{A}$, there exists a unique symmetric positive definite matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$). Suppose $\mathbf{F} = \mathbf{R}\mathbf{U}$ where $\mathbf{R} =$ orthogonal, $\mathbf{U} =$ symmetric positive definite. Then

$$\mathbf{F}^T\mathbf{F} = \mathbf{U}^T\mathbf{R}^T\mathbf{R}\mathbf{U} = \mathbf{U}^T\mathbf{U} = \mathbf{U}^2$$

Thus $\mathbf{U}$ can be the unique matrix whose square is the symmetric positive definite matrix $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. Then set $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$, since

$$\mathbf{R}^T\mathbf{R} = \mathbf{U}^{-T}\mathbf{F}^T\mathbf{F}\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{C}\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{U}^{-2}\mathbf{U}^{-1} = \mathbf{I}$$

Similarly, we prove that $\mathbf{F}$ can be written

$$\mathbf{F} = \mathbf{V}\mathbf{S}$$

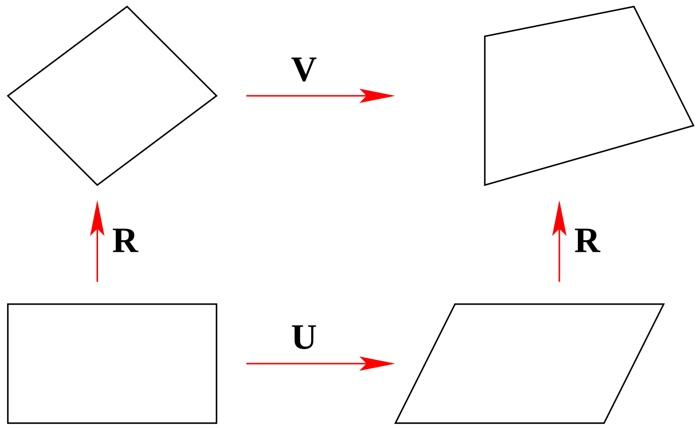

Figure 1.7: The Polar Decomposition Theorem.

where $\mathbf{V}$ is symmetric positive definite and $\mathbf{S}$ is an orthogonal matrix. One can then show that $\mathbf{S} = \mathbf{R}$. (Exercise) □

Thus, if $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ are the right and left Cauchy-Green deformation tensors, and

$$\mathbf{F} = \mathbf{R}\mathbf{U} \sim \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$$

then

$$\boxed{\begin{aligned} &\mathbf{C} = \mathbf{U}^T\mathbf{R}^T\mathbf{R}\mathbf{U} = \mathbf{U}^T\mathbf{U} = \mathbf{U}^2 \\ &\mathbf{B} = \mathbf{V}\mathbf{R}\mathbf{R}^T\mathbf{V} = \mathbf{V}\mathbf{V}^T = \mathbf{V}^2 \end{aligned}}$$

where $\mathbf{U}$ and $\mathbf{V}$ are the right and left stretch tensors, respectively.

Clearly, the Polar Decomposition Theorem establishes that the deformation gradient $\mathbf{F}$ can be obtained (or can be viewed as the result of) a stretching followed by a rotation or vice versa (see Fig. 1.7).

1.8 Principal Directions and Invariants of Deformation and Strain

For a given deformation tensor field $\mathbf{C}(\mathbf{X})$ and strain field $\mathbf{E}(\mathbf{X})$ (at point $\mathbf{X}$), recall that $d\mathbf{X}^T\mathbf{C}d\mathbf{X} = 2d\mathbf{X}^T\mathbf{E}d\mathbf{X} - d\mathbf{X}^Td\mathbf{X}$ is the square dS^2 of a material line segment in the current configuration. Then a measure of the stretch or compression of a unit material element originally oriented along a unit vector $\mathbf{m}$ is given by

$$\Delta(\mathbf{m}) = dS^2 - 1 = 2\mathbf{m}^T\mathbf{E}\mathbf{m}$$
$$\mathbf{m} \cdot \mathbf{m} = \text{“}\mathbf{m}^T\mathbf{m}\text{”} = 1$$

One may ask: of all possible directions $\mathbf{m}$ at $\mathbf{X}$, which choice results in the largest (or smallest) value of $\Delta(\mathbf{m})$?

This is a constrained maximization/minimization problem: find $\mathbf{m} = \mathbf{m}_{max}$ (or $\mathbf{m}_{min}$) that makes $\Delta(\mathbf{m})$ as large (or small as possible, subject to the constraint $\mathbf{m}^T\mathbf{m} = 1$. To resolve this problem, we use the method of Lagrange multipliers. Denote by $L(m, \lambda) = \Delta(\mathbf{m}) - \lambda(\mathbf{m}^T\mathbf{m} - 1)$, λ being the Lagrange multiplier. The maxima (on minimize and maximize points) of L satisfy,

$$\frac{\partial L(\mathbf{m}, \lambda)}{\partial m} = 0 = 4(E\mathbf{m} - \lambda\mathbf{m})$$

Thus, unit vectors $\mathbf{m}$ that maximize or minimize $\Delta(\mathbf{m})$ are associated with multipliers λ and satisfy

$$\mathbf{E}\mathbf{m} = \lambda\mathbf{m}, \qquad \mathbf{m}^T\mathbf{m} = 1$$

That is, $(\mathbf{m}, \lambda)$ are eigenvector/eigenvalue pairs of the strain tensor $\mathbf{E}$, and $\mathbf{m}$ is normalized so that $\mathbf{m}^T\mathbf{m} = 1$ (or $\|\mathbf{m}\| = 1$).

The following fundamental properties of the above eigenvalue problem can be listed.

1. There are three real eigenvalues and three eigenvectors of $\mathbf{E}$ (at $\mathbf{X}$); we adopt the ordering $\lambda_1 \geq \lambda_2 \geq \lambda_3$

2. For $\lambda_i \neq \lambda_j$, the corresponding eigenvectors are orthogonal (for pairs $(\mathbf{m}_i, \lambda_i)$ and $(\mathbf{m}_j, \lambda_j)$, $\mathbf{m}_i^T\mathbf{m}_j = \delta_{ij}$, as can be seen as follows:

$$\mathbf{m}_i^T(\lambda_j\mathbf{m}_j) = \mathbf{m}_i^T C\mathbf{m}_j = \mathbf{m}_j^T C\mathbf{m}_i = \mathbf{m}_j^T(\lambda_i\mathbf{m}_i)$$

so

$$(\lambda_i - \lambda_j)\mathbf{m}_i^T\mathbf{m}_j = 0$$

so

$$\text{if } \lambda_i \neq \lambda_j, \qquad \mathbf{m}_i^T\mathbf{m}_j = \delta_{ij}, \quad 1 \leq i, j \leq 3$$

(if $\lambda_i = \lambda_j$, we can always construct $\mathbf{m}_j$ so that it is orthogonal to $\mathbf{m}_i$)

3. Let $\mathbf{N}$ be the matrix with the mutually orthogonal eigenvectors as rows. Then

$$\mathbf{N}^T\mathbf{C}\ \mathbf{N} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \text{diag}\ \{\lambda_i, i = 1, 2, 3\}$$

The coordinate system defined by the mutually orthogonal triad of eigenvectors define the principal directions and values of $\mathbf{C}$ at $\mathbf{X}$. For this choice of a basis,

$$\boxed{\mathbf{E} = \sum_{i=1}^{3} \lambda_i \mathbf{m}_i \otimes \mathbf{m}_i}$$

4. If $\lambda_1 \geq \lambda_2 \geq \lambda_3$, λ_1 corresponds to the maximum, λ_3 to the minimum, and λ_2 to a "mini-max" principal value of $\mathbf{E}$ (or of $\Delta(\mathbf{n})$).

5. The *characteristic polynomial* of $\mathbf{E}$ is

$$\det(\mathbf{E} - \lambda\mathbf{I}) = -\lambda^3 + I(\mathbf{E})\lambda^2 - I\!I(\mathbf{E})\lambda + I\!I\!I(\mathbf{E})$$

   where I, $I\!I$, $I\!I\!I$ are the *principal invariants* of $\mathbf{E}$:

$$\begin{aligned} I(\mathbf{E}) &= \text{trace } \mathbf{E} \equiv \text{tr } \mathbf{E} = \mathbf{E}_{ii} = \mathbf{E}_{11} + \mathbf{E}_{22} + \mathbf{E}_{33} \\ &= \lambda_1 + \lambda_2 + \lambda_3 \end{aligned}$$

$$\begin{aligned} I\!I(\mathbf{E}) &= \frac{1}{2}(\text{tr } \mathbf{E})^2 - \frac{1}{2}\text{tr}\mathbf{E}^2 \\ &= \text{tr Cof } \mathbf{E} \\ &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 \end{aligned}$$

$$\begin{aligned} I\!I\!I(\mathbf{E}) &= \det \mathbf{E} \\ &= \frac{1}{6}\{(\text{tr } \mathbf{E})^3 - 3 \text{ tr } \mathbf{E} \text{ tr } \mathbf{E}^2 + 2 \text{ tr } \mathbf{E}^3\} \\ &= \lambda_1\lambda_2\lambda_3 \end{aligned}$$

   (An invariant of a real matrix $\mathbf{C}$ is any real-valued function $\mu(\mathbf{C})$ with the property $\mu(\mathbf{C}) = \mu(\mathbf{A}^{-1}\mathbf{C}\mathbf{A})$ for all invertible matrices $\mathbf{A}$).

Chapter 2

Mass and Momentum

Mass: the property of a body that is a measure of the amount of material it contains and causes it to have weight in a gravitational field.

In continuum mechanics, the mass of a body is continuously distributed over its volume and is an integral of a density field $\varrho : \bar{\Omega}_0 \to \mathbb{R}^+$ called the mass density. The total mass $\mathcal{M}(\mathcal{B})$ of a body is independent of the motion $\boldsymbol{\varphi}$, but the mass density ϱ can, of course, change as the volume of the body changes while in motion. Symbolically,

$$\mathcal{M}(\mathcal{B}) = \int_{\Omega_t} \varrho \, dx$$

where dx = volume element in the current configuration Ω_t of the body.

Given two motions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ (see Figure 2.1), let $\varrho_{\boldsymbol{\varphi}}$ and ϱ_{ψ} denote the mass densities in the configurations $\boldsymbol{\varphi}(\Omega_0)$ and $\boldsymbol{\psi}(\Omega_0)$, respectively. Since the total mass is independent of the motion,

$$\boxed{\mathcal{M}(\mathcal{B}) = \int_{\boldsymbol{\varphi}(\Omega_0)} \varrho_{\boldsymbol{\varphi}} \, dx = \int_{\boldsymbol{\psi}(\Omega_0)} \varrho_{\boldsymbol{\psi}} \, dx}$$

This fact represents the principle of *conservation of mass*. The mass of a body $\mathcal{B}$ is thus an invariant property (measuring the amount of material in $\mathcal{B}$); the weight of $\mathcal{B}$ is defined as $g\mathcal{M}(\mathcal{B})$ where g is a constant gravity field. Thus, a body may weigh differently in different gravity fields (e.g. the earth's gravity as opposed to that on the moon), but its mass is the same.

2.1 Local Forms of the Principle of Conservation of Mass

Let $\varrho_0 = \varrho_0(\mathbf{X})$ be the mass density of a body in its reference configuration and $\varrho = \varrho(\boldsymbol{x}, t)$ the mass density in the current configuration Ω_t. Then

$$\int_{\Omega_0} \varrho_0(\mathbf{X}) \, dX = \int_{\Omega_t} \varrho(\boldsymbol{x}) \, dx$$

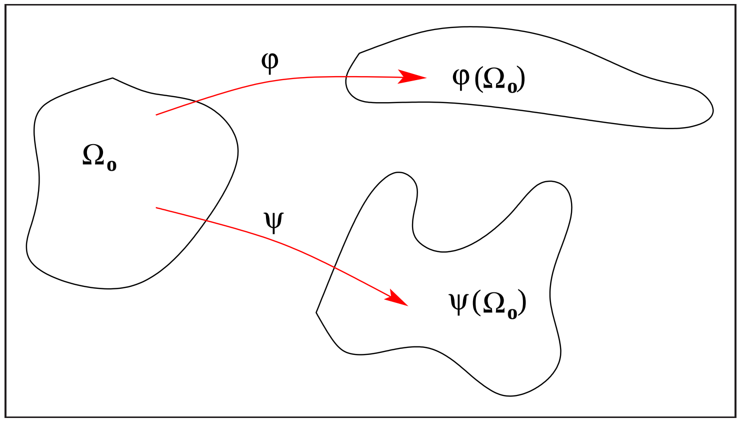

Figure 2.1: Two motions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$.

(where the dependence of ϱ on t has been suppressed). But $dx = \det \mathbf{F}(\mathbf{X})\, dX$, so

$$\int_{\Omega_0} [\varrho_0(\mathbf{X}) - \varrho(\boldsymbol{\varphi}(\mathbf{X})) \det \mathbf{F}(\mathbf{X})]\, dX = 0$$

and, therefore

$$\boxed{\varrho_0(\mathbf{X}) = \varrho(\boldsymbol{x}) \det \mathbf{F}(\mathbf{X})}$$

This is the *material description* (or the *Lagrangian formulation*) of the principle of conservation of mass. To obtain the *spatial description* (or *Eulerian formulation*), we observe that the invariance of total mass can be expressed as:

$$\frac{d}{dt} \int_{\Omega_t} \varrho(\boldsymbol{x}, t)\, dx = 0$$

Changing to the material coordinates gives

$$0 = \frac{d}{dt} \int_{\Omega_0} \varrho(\boldsymbol{x}, t) \det \mathbf{F}(\mathbf{X}, t)\, dX = \int_{\Omega_0} (\varrho\, \dot{\overline{\det \mathbf{F}}} + \dot{\varrho} \det \mathbf{F})\, dX$$

where $\dot{(\cdot)} = d(\cdot)/dt$. Recalling that $\dot{\overline{\det \mathbf{F}}} = \det \mathbf{F}\ \mathrm{div}\ \mathbf{v}$, we have

$$0 = \int_{\Omega_0} \det \mathbf{F} \left(\varrho\ \mathrm{div}\ \mathbf{v} + \frac{\partial \varrho}{\partial t} + \mathbf{v} \cdot \mathrm{grad}\ \varrho \right)\, dX$$

from which we conclude:

$$\boxed{\frac{\partial \varrho}{\partial t} + \mathrm{div}\ (\varrho \mathbf{v}) = 0}$$

2.2 Momentum

The momentum of a material body is a property the body has by virtue of its mass and its velocity. Given a motion $\boldsymbol{\varphi}$ of a body $\mathcal{B}$ of mass density ϱ, the *linear momentum* $I(\mathcal{B}, t)$ of $\mathcal{B}$ at time t and the *angular momentum* $H(\mathcal{B}, t)$ of $\mathcal{B}$ at time t about the origin $\mathbf{0}$ of the spatial coordinate system are defined by

$$\boxed{\begin{aligned} I(\mathcal{B}, t) &= \int_{\Omega_t} \varrho \mathbf{v} \; dx \\ H(\mathcal{B}, t) &= \int_{\Omega_t} \boldsymbol{x} \times \varrho \mathbf{v} \; dx \end{aligned}}$$

Again, dx ($= dx_1 dx_2 dx_3$) is the volume element in Ω_t.

The rates of change of momenta (both I and H) are of fundamental importance. To calculate rates, first notice that for any smooth field $w = w(\boldsymbol{x}, t)$,

$$\frac{d}{dt} \int_{\Omega_t} w \varrho \, dx = \frac{d}{dt} \int_{\Omega_0} w(\boldsymbol{\varphi}(\mathbf{X}, t), t) \varrho(\boldsymbol{x}, t) \det \mathbf{F}(\mathbf{X}, t) \; dX = \int_{\Omega_0} \frac{dw}{dt} \varrho_0 \; dX = \int_{\Omega_t} \frac{dw}{dt} \varrho \; dx$$

Thus,

$$\boxed{\begin{aligned} \frac{dI(\mathcal{B}, t)}{dt} &= \int_{\Omega_t} \varrho \frac{d\mathbf{v}}{dt} \; dx \\ \frac{dH(\mathcal{B}, t)}{dt} &= \int_{\Omega_t} \boldsymbol{x} \times \varrho \frac{d\mathbf{v}}{dt} \; dx \end{aligned}}$$

Chapter 3

Force and Stress in Deformable Bodies

The concept of force is used to characterize the interaction of the motion of a material body with its environment. More generally, as will be seen later, force is a characterization of interactions of the body with agents that cause a change in its momentum. In continuum mechanics, there are basically two types of forces: 1) *contact forces*, representing the contact of the boundary surfaces of the body with the exterior universe, i.e. its exterior environment, or the contact of internal parts of the body on surfaces that separate them, and 2) *body forces*, acting on material points of the body by its environment.

Body Forces. Examples of body forces are the weight-per-unit volume exerted by the body by gravity or forces per unit volume exerted by an external magnetic field. Body forces are a type of *external force*, naturally characterized by a given vector-valued field $\mathbf{f}$ called the *body force density per unit volume*. The total body force is then

$$\int_{\Omega_t} \mathbf{f}(\boldsymbol{x}, t)\ dx$$

Alternatively, we can measure the body force with a density $\mathbf{b}$ per unit mass: $\mathbf{f} = \varrho\mathbf{b}$. Then

$$\int_{\Omega_t} \mathbf{f}\ dx = \int_{\Omega_t} \varrho\mathbf{b}\ dx.$$

Contact Forces. Contact forces are also called *surface forces* because the contact of one body with another or with its surroundings must take place on a material surface. Contact forces fall into two categories:

1. *External contact forces* representing the contact of the exterior boundary surface of the body with the environment outside the body, and

2. *Internal contact forces* representing the contact of arbitrary parts of the body that touch one another on parts of internal surfaces they share on their common boundary.

The Concept of Stress. There is essentially no difference between the structure of external or internal contact forces; they differ only in what is *interpreted* as the boundary

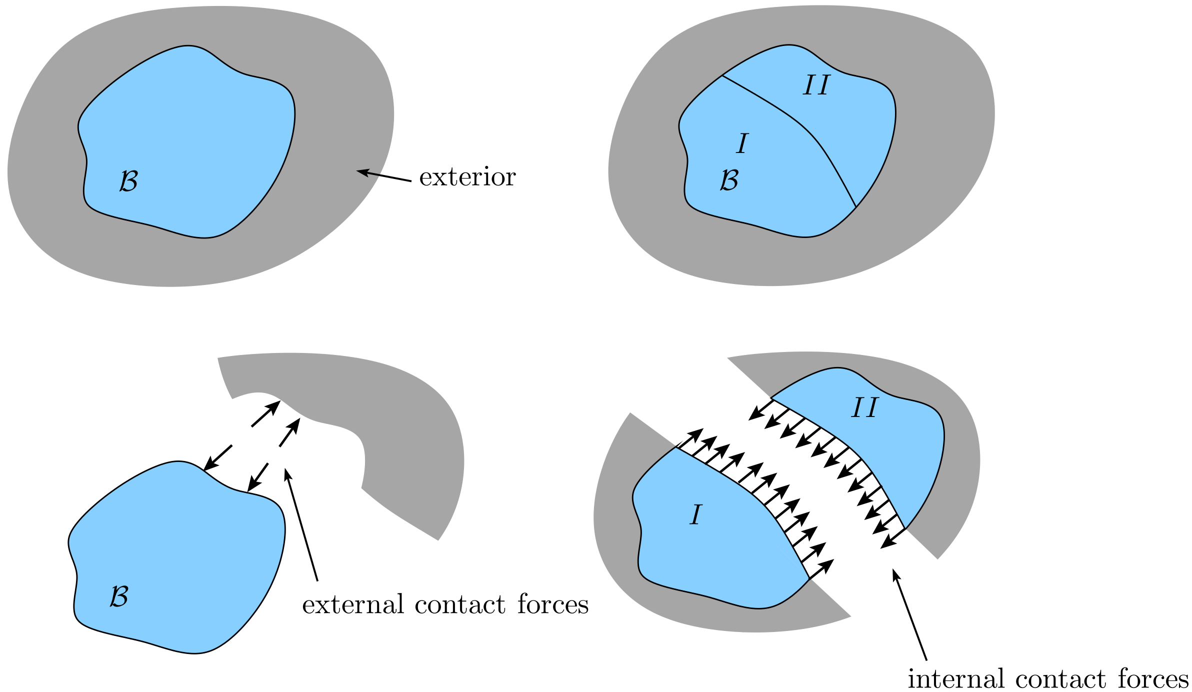

Figure 3.1: External and internal contact forces.

that separates a material body from its surroundings. Portion I of the partitioned body in Fig. 3.1 could just as well been defined as body $\mathcal{B}$ and portion II would then be part of its exterior environment.

Fig. 3.2 is an illustration of the discrete version of the various forces: a collection of rigid spherical balls of weight $\mathbf{W}$ each resting in a rigid bowl and pushed downward by balancing a book on the top ball of weight $\mathbf{P}$. Explode the collection of balls into free bodies as shown. The five balls are the body $\mathcal{B}$. The exterior contact with the outside environment is represented by the force P and the contact forces N representing the fact that the balls press against the bowl and the bowl against the balls in an equal and opposite way. Then $\mathbf{P}$, $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ are external contact forces. The weights $\mathbf{W}$ are the body forces. Internally, the balls touch one another on exterior surfaces of each ball. The action of a given ball on another, is equal and opposite to the action of the other balls on the given balls. These contact forces are internal. They cancel out (balance), when the balls are reassembled into the whole body $\mathcal{B}$.

In the case of a continuous body, the same idea applies, except that the contact of any part of the body (part I say) with the complement (part II) is continuous (as there are now a continuum of material particles in contact along the contact surface) and the nature of these contact forces depends upon how (we visualize) the body is partitioned. Thus, at a point $\boldsymbol{x}$, if we separate $\mathcal{B}$ (conceptually) into bodies I and II with a surface AA defined with an orientation given by a unit vector $\mathbf{n}$, the distribution of contact forces at a point $\boldsymbol{x}$ on the surface will be quite different than that produced by a different partitioning of the body

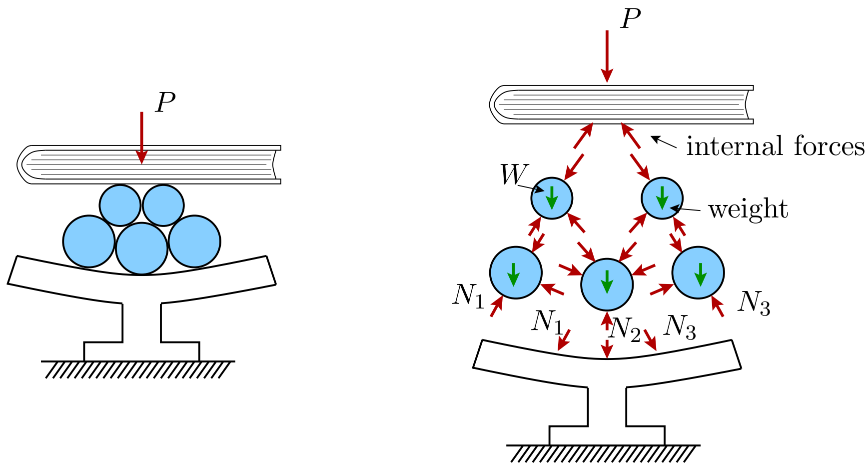

Figure 3.2: Illustrative example of the stress concept.

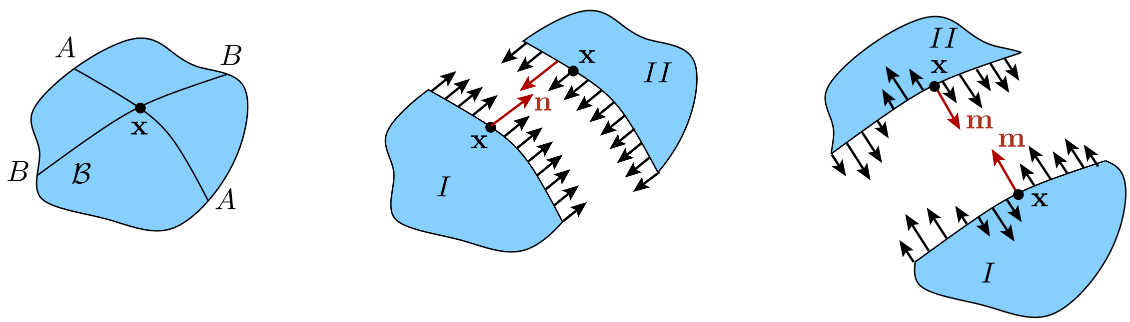

Figure 3.3: The Cauchy hypothesis.

defined by a different surface BB though the same point $\boldsymbol{x}$ but with orientation defined by a different unit vector $\mathbf{m}$ (see Fig. 3.3).

These various possibilities are captured by the so-called *Cauchy hypothesis*: there exists a vector-valued surface (contact) force density

$$\boldsymbol{\sigma}(\mathbf{n}, \boldsymbol{x}, t)$$

giving the force per unit area on an oriented surface Γ through $\boldsymbol{x}$ with unit normal $\mathbf{n}$, at time t. The convention is that $\boldsymbol{\sigma}(\mathbf{n}, \boldsymbol{x}, t)$ defines the force per unit area on the "negative" side of the material ($\mathbf{n}$ is a unit exterior or outward normal) exerted by the material on the opposite side (thus, the direction of $\boldsymbol{\sigma}$ on body II is opposite to that on I because the exterior normals are in opposite directions – see Figure 3.4).

Thus, if the vector field $\boldsymbol{\sigma}(\mathbf{n}, \boldsymbol{x}, t)$ were known, one could pick an arbitrary point $\boldsymbol{x}$ in the body (or, equivalently, in the current configuration Ω_t) at time t, and pass a surface through $\boldsymbol{x}$ with orientation given by the unit normal $\mathbf{n}$. The vector $\boldsymbol{\sigma}(\mathbf{n}, \boldsymbol{x}, t)$ would then represent the contact force per unit area on this surface at point $\boldsymbol{x}$ at time t. The surface Γ through $\boldsymbol{x}$ partitions the body into two parts: the orientation of the vector $\boldsymbol{\sigma}(\mathbf{n}, \boldsymbol{x}, t)$ on

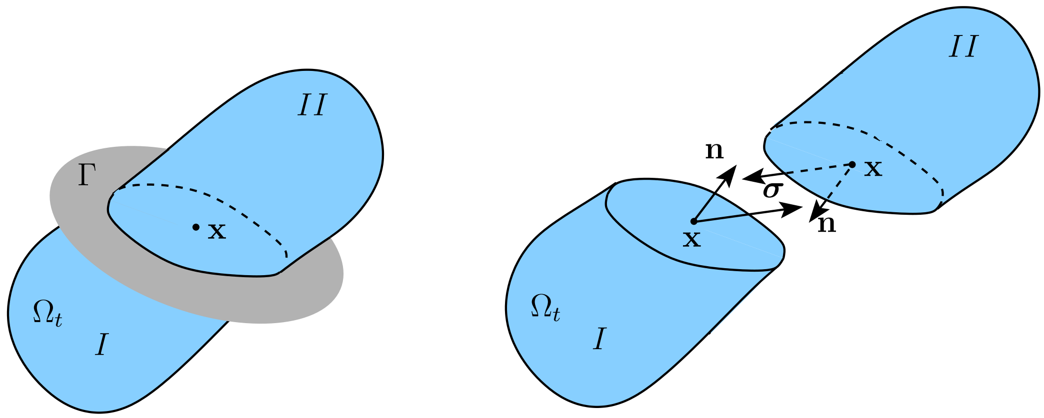

Figure 3.4: The stress vector.

one part (at $\boldsymbol{x}$) is opposite to that on the other part. The vector field $\boldsymbol{\sigma}$ is called the *stress vector field* and $\boldsymbol{\sigma}(\mathbf{n}, \boldsymbol{x}, t)$ is the *stress vector* at $\boldsymbol{x}$ and t for orientation $\mathbf{n}$. The total force on surface Γ is

$$\int_\Gamma \boldsymbol{\sigma}(\mathbf{n}, \boldsymbol{x}, t)\; dA,$$

dA being the surface area element.

The total force acting on body $\mathcal{B}$ and total moment about the origin $\mathbf{0}$, at time t when the body occupies the current configuration Ω_t, are, respectively,

$$\boxed{\begin{aligned} \boldsymbol{\mathcal{F}}(\mathcal{B}, t) &= \int_{\Omega_t} \mathbf{f}\; dx + \int_{\partial\Omega_t} \boldsymbol{\sigma}(\mathbf{n})\; dA \\ \mathbf{M}(\mathcal{B}, t) &= \int_{\Omega_t} \boldsymbol{x} \times \mathbf{f}\; dx + \int_{\partial\Omega_t} \boldsymbol{x} \times \boldsymbol{\sigma}(\mathbf{n})\; dA \end{aligned}}$$

where we have suppressed the dependence of $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\sigma}$ on $\boldsymbol{x}$ and t.

Chapter 4

The Principles of Balance of Linear and Angular Momentum

The momentum balance laws are the fundamental axioms of mechanics that connect motion and force:

The Principle of Balance of Linear Momentum

The time-rate-of-change of linear momentum $I(\mathcal{B}, t)$ *of a body* $\mathcal{B}$ *at time* t *equals (or is balanced by) the total force* $\boldsymbol{\mathcal{F}}(\mathcal{B}, t)$ *acting on the body*:

$$\frac{dI(\mathcal{B}, t)}{dt} = \boldsymbol{\mathcal{F}}(\mathcal{B}, t)$$

The Principle of Balance of Angular Momentum

The time-rate-of-change of angular momentum $H(\mathcal{B}, t)$ *of a body* $\mathcal{B}$ *at time* t *equals (or is balanced by) the total force* $\mathbf{M}(\mathcal{B}, t)$ *acting on the body*:

$$\frac{dH(\mathcal{B}, t)}{dt} = \mathbf{M}(\mathcal{B}, t)$$

Thus, for a continuous media

$$\int_{\Omega_t} \varrho \frac{d\mathbf{v}}{dt}\, dx = \int_{\Omega_t} \mathbf{f}\, dx + \int_{\partial\Omega_t} \boldsymbol{\sigma}(\mathbf{n})\ dA$$

$$\int_{\Omega_t} \boldsymbol{x} \times \varrho \frac{d\mathbf{v}}{dt}\ dx = \int_{\Omega_t} \boldsymbol{x} \times \mathbf{f}\ dx + \int_{\partial\Omega_t} \boldsymbol{x} \times \boldsymbol{\sigma}(\mathbf{n})\ dA$$

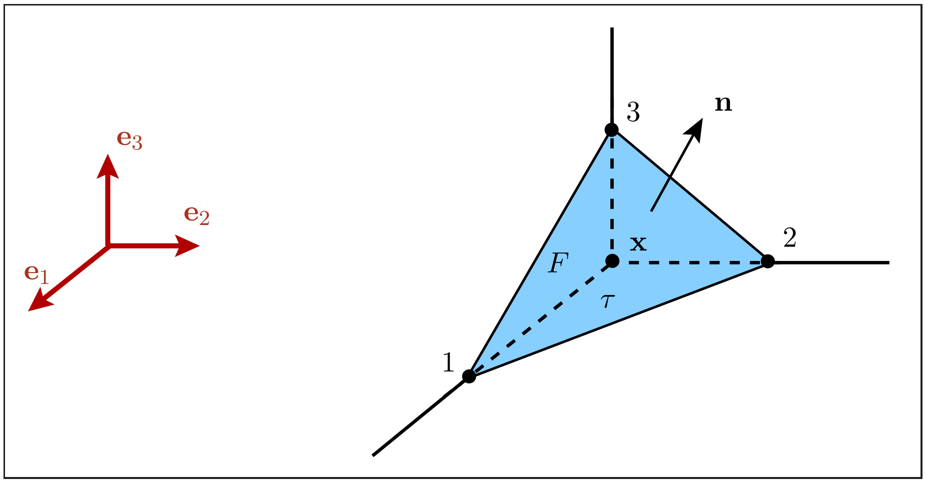

Figure 4.1: Tetrahedron τ for the proof of Cauchy's Theorem.

4.1 Cauchy's Theorem: The Cauchy Stress Tensor

Theorem: *At each time t, let the body force density $\mathbf{f} : \Omega_t \to \mathbb{R}^3$ be a continuous function of $\boldsymbol{x}$ and the stress vector field $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{n}, \boldsymbol{x}, t)$ be continuously differentiable with respect to $\boldsymbol{n}$ for each $\boldsymbol{x} \in \Omega_t$ and continuously differentiable with respect to $\boldsymbol{x}$ for each $\boldsymbol{n}$. Then the principles of balance of linear and angular momentum imply that there exists a continuously differential tensor field $\boldsymbol{T} : \bar{\Omega}_t \to \mathbb{M}^3$ (the set of square matrices of order three) such that*

$$\boldsymbol{\sigma}(\mathbf{n}, \boldsymbol{x}, t) = \mathbf{T}(\boldsymbol{x}, t)\mathbf{n}, \qquad \forall \boldsymbol{x} \in \bar{\Omega}_t, \ \forall \mathbf{n}$$

and

$$\mathbf{T}(\boldsymbol{x}, t) = \mathbf{T}(\boldsymbol{x}, t)^T, \qquad \forall \boldsymbol{x} \in \bar{\Omega}_t$$

Proof of Cauchy's Theorem. The proof is classical: pick $\boldsymbol{x} \in \Omega_t$. Since Ω_t is open, we can construct a tetrahedron τ with "vertex" at $\boldsymbol{x}$ with three faces parallel to the x_i-coordinate planes and with an exterior face F with unit normal $\mathbf{n} = n_i \ \mathbf{e}_i$, $n_i > 0$ (see Fig. 4.1).

The faces of τ opposite to the vertices i, $i = 1, 2, 3$, are denoted F_i and $\text{area}(F_i) = n_i\text{area}(F)$. According to the principle of balance of linear momentum,

$$\int_\tau \mathbf{f} \, dx + \int_{\partial\tau} \boldsymbol{\sigma}(\mathbf{n}) \, dA - \int_\tau \varrho \frac{d\mathbf{v}}{dt} \, dx = \mathbf{0}$$

Let $\mathbf{f} = f_i \mathbf{e}_i$, $\boldsymbol{\sigma}(\mathbf{n}) = \sigma_i(\mathbf{n})\mathbf{e}_i$, $\mathbf{v} = v_i \mathbf{e}_i$. Then, for each component of the above equation, and using the mean-value theorem,

$$\int_{\partial\tau} \sigma_i(\mathbf{n}, \boldsymbol{x})\ dA = \int_{F_j = F n_j} \sigma_i(-\mathbf{e}_j, \boldsymbol{x})\ dA + \int_F \sigma_i(\mathbf{n}, \boldsymbol{x})\ dA$$
$$= \sigma_i(-\mathbf{e}_j, \boldsymbol{x}_j^*)\ n_j\ \text{area}(F) + \sigma_i(\mathbf{n}, \widehat{\boldsymbol{x}})\ \text{area}(F)$$

for $\boldsymbol{x}_j^* \in F_j$ and $\widehat{\boldsymbol{x}} \in F$, so that

$$\int_{\partial\tau} \sigma_i(\mathbf{n}, \boldsymbol{x})\ dA = \int_\tau \left(\varrho \frac{dv_i(\boldsymbol{x})}{dt} - f_i(\boldsymbol{x}) \right)\ dx, \qquad i = 1, 2, 3$$

Thus, since volume$(\tau) = C$ area$(F)^{3/2}$, C a constant and $i = 1, 2, 3$,

$$\left| n_j \sigma_i(-\mathbf{e}_j, \boldsymbol{x}_j^*) + \sigma_i(\mathbf{n}, \widehat{\boldsymbol{x}}) \right| \text{area}(F) \le C \sup_{\boldsymbol{x} \in \tau} \left| \varrho(x) \frac{dv_i(\boldsymbol{x})}{dt} - f_i(\boldsymbol{x}) \right| \times \text{area}(F)^{3/2}$$

Keeping $\mathbf{n}$ fixed, we shrink the tetrahedron τ to the vertex $\boldsymbol{x}$ by collapsing the vertices to $\boldsymbol{x}$ (area$(F) \to 0$), and obtain

$$\sigma_i(\mathbf{n}, \boldsymbol{x}) = -n_j \sigma_i(-\mathbf{e}_j, \boldsymbol{x}), \qquad 1 \le i, j \le 3$$

or, since $\boldsymbol{\sigma}(\mathbf{n}) = \sigma_i(\mathbf{n})\mathbf{e}_i$,

$$\boxed{\boldsymbol{\sigma}(\mathbf{n}, \boldsymbol{x}) = -n_j \boldsymbol{\sigma}(-\mathbf{e}_j, \boldsymbol{x})}$$

Now, for each vector $\boldsymbol{\sigma}(\mathbf{e}_j, \boldsymbol{x})$, define functions $T_{ij}(\boldsymbol{x})$ such that

$$\boldsymbol{\sigma}(\mathbf{e}_j, \boldsymbol{x}) = T_{ij}(\boldsymbol{x})\mathbf{e}_i$$

Then

$$\boxed{\sigma_i(\mathbf{n}, \boldsymbol{x}) = T_{ij}(\boldsymbol{x}) n_j, \quad \text{or} \quad \boldsymbol{\sigma}(\mathbf{n}, \boldsymbol{x}) = \mathbf{T}(\boldsymbol{x})\mathbf{n}}$$

as asserted (by continuity, these hold for all $\boldsymbol{x} \in \bar{\Omega}_t$). The tensor $\mathbf{T}$ is of course the Cauchy stress tensor. We will take up the proof that $\mathbf{T}$ is symmetric later (as an exercise), which follows from the principle of balance of angular momentum. □

4.2 The Equations of Motion (Linear Momentum)

According to the divergence theorem,

$$\int_{\partial\Omega_t} \mathbf{T}\mathbf{n}\, dA = \int_{\Omega_t} \text{div}\ \mathbf{T}\, dx$$

Thus,

$$\int_{\partial\Omega_t} \boldsymbol{\sigma}(\mathbf{n})\, dA = \int_{\partial\Omega_t} \mathbf{T}\mathbf{n}\, dA = \int_{\Omega_t} \text{div}\ \mathbf{T}\, dx$$

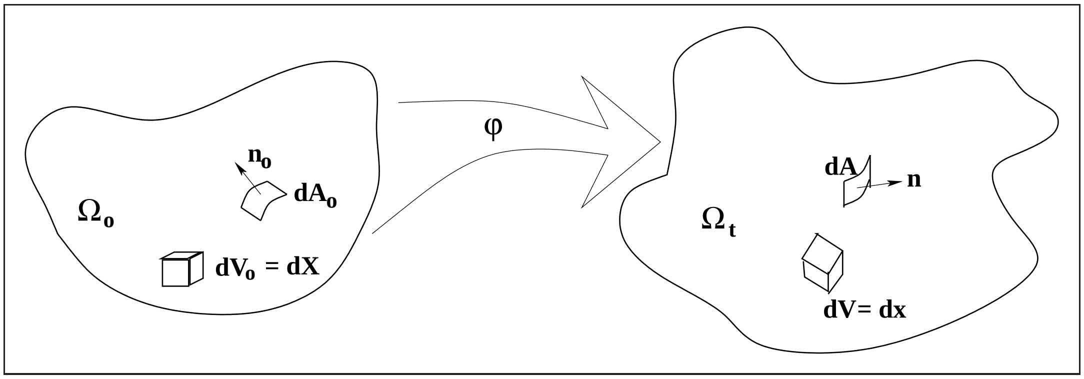

Figure 4.2: Mapping of volume and surface elements.

It follows that the principle of balance of linear momentum can be written,

$$\int_{\Omega_t} \varrho \frac{d\mathbf{v}}{dt} dx = \int_{\Omega_t} (\mathbf{f} + \text{div } \mathbf{T}) dx$$

where $\mathbf{T}$ is the Cauchy stress tensor. Thus

$$\int_{\Omega_t} (\text{div } \mathbf{T} + \mathbf{f} - \varrho \frac{d\mathbf{v}}{dt}) dx = 0$$

But this must also hold for any arbitrary subdomain $G \subset \Omega_t$. Thus,

$$\boxed{\text{div } \mathbf{T} + \mathbf{f} = \varrho \frac{d\mathbf{v}}{dt}}$$

or

$$\text{div } \mathbf{T}(\boldsymbol{x}, t) + \mathbf{f}(\boldsymbol{x}, t) = \varrho(\boldsymbol{x}, t) \frac{d\mathbf{v}}{dt}(\boldsymbol{x}, t)$$

Returning to the proof of Cauchy's Theorem, we apply the principle of balance of angular momentum to the tetrahedron and use the fact that div $\mathbf{T} + \mathbf{f} - \varrho d\mathbf{v}/dt = 0$. This leads to the conclusion that

$$\mathbf{T}^T = \mathbf{T}$$

Details are left as an exercise.

4.3 The Equations of Motion Referred to the Reference Configuration: The Piola-Kirchhoff Stress Tensors

In the current configuration,

$$\text{div } \mathbf{T}(\boldsymbol{x}) + \mathbf{f}(\boldsymbol{x}) = \varrho(\boldsymbol{x}) \left(\frac{\partial \mathbf{v}(\boldsymbol{x})}{\partial t} + \mathbf{v}(\boldsymbol{x}) \cdot \text{grad } \mathbf{v}(\boldsymbol{x}) \right)$$

where we have again not expressed the dependence on time t for simplicity. Here div and grad are defined with respect to $\boldsymbol{x}$; i.e. the dependent variables are regarded as functions of spatial position $\boldsymbol{x}$ (and t). We now refer the fields to the reference configuration:

$$\begin{aligned}
\mathbf{f}_0(\mathbf{X}) &= \mathbf{f}(\boldsymbol{x}) \det \mathbf{F}(\mathbf{X}) \qquad (\boldsymbol{x} = \boldsymbol{\varphi}(\mathbf{X})) \\
\varrho_0(\mathbf{X}) &= \varrho(\boldsymbol{x}) \det \mathbf{F}(\mathbf{X}) \\
\mathbf{P}(\mathbf{X}) &= \det \mathbf{F}(\mathbf{X})\ \mathbf{T}(\boldsymbol{x})\ \mathbf{F}(\mathbf{X})^{-T} \\
&= \mathbf{T}(\boldsymbol{x})\ \mathrm{Cof}\ \mathbf{F}(\mathbf{X}) \qquad \text{(by the Piola transformation)}
\end{aligned}$$

The tensor $\mathbf{P}(\mathbf{X})$ is called the *First Piola-Kirchhoff Stress Tensor*. Note that $\mathbf{P}$ is not symmetric; however $\mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T$ since $\mathbf{T}$ is symmetric. Recalling earlier proof, we have

$$\begin{aligned}
\frac{\partial}{\partial X_j} P_{ij} = \frac{\partial}{\partial X_j}(\mathbf{T}\,\mathrm{Cof}\,(\mathbf{F})) &= \det \mathbf{F}\ \frac{\partial}{\partial X_\ell} T_{ik} \cdot \frac{\partial x_\ell}{\partial X_j} \cdot \mathbf{F}_{kj}^{-T} + T_{ik} \cancelto{0}{\frac{\partial}{\partial X_j}((\mathrm{Cof}\,\mathbf{F})_{kj})} \\
&= \det \mathbf{F}\ \frac{\partial}{\partial x_k} T_{ik} = \det \mathbf{F}\ \mathrm{div}\ \mathbf{T}
\end{aligned}$$

i.e.

$$\boxed{\mathrm{Div}\ \mathbf{P} = \det \mathbf{F}\ \mathrm{div}\ \mathbf{T}}$$

so

$$\mathrm{div}\,\mathbf{T}(\boldsymbol{x}) + \mathbf{f}(\boldsymbol{x}) = \frac{1}{\det \mathbf{F}}(\mathrm{Div}\,\mathbf{P} + \mathbf{f}_0)(\mathbf{X}) = \varrho \frac{d\mathbf{v}}{dt}(\boldsymbol{x}) = \varrho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2}(\boldsymbol{x}) \frac{1}{\det \mathbf{F}}$$

Thus, the equations of motion (linear and angular momentum) referred to the reference configuration are:

$$\boxed{\begin{aligned}
\mathrm{Div}\ \mathbf{P}(\mathbf{X}) + \mathbf{f}_0(\mathbf{X}) &= \varrho_0(\mathbf{X})\ddot{\mathbf{u}}(\mathbf{X}) \\
\mathbf{P}(\mathbf{X})\ \mathbf{F}^T(\mathbf{X}) &= \mathbf{F}(\mathbf{X})\ \mathbf{P}^T(\mathbf{X})
\end{aligned}}$$

Here the dependence of $\mathbf{P}$, $\mathbf{f}_0$, $\ddot{\mathbf{u}}$, and $\mathbf{F}$ on time t is not indicated for simplicity; but note that ϱ_0 is independent of t. Alternatively, these equations can be written in terms of a symmetric tensor:

$$\mathbf{S}(\mathbf{X}) = \mathbf{S}(\mathbf{X}, t) = \mathbf{F}(\mathbf{X})^{-1}\mathbf{P}(\mathbf{X}) = \det \mathbf{F}\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} = \mathbf{F}^{-1}\mathbf{T}\,\mathrm{Cof}\,\mathbf{F}$$

The tensor $\mathbf{S}$ is the so-called *Second Piola-Kirchhoff Stress Tensor*. Then

$$\boxed{\begin{aligned}
\mathrm{Div}\ \mathbf{F}(\mathbf{X})\mathbf{S}(\mathbf{X}) + \mathbf{f}_0(\mathbf{X}) &= \varrho_0(\mathbf{X})\ddot{\mathbf{u}}(\mathbf{X}) \\
\mathbf{S}(\mathbf{X}) &= \mathbf{S}(\mathbf{X})^T
\end{aligned}}$$

In summary

Cauchy Stress:	$\mathbf{T}$	$= (\det \mathbf{F})^{-1}\ \mathbf{P}\ \mathbf{F}^T$ $= (\det \mathbf{F})^{-1}\ \mathbf{F}\ \mathbf{S}\ \mathbf{F}^T$
First Piola-Kirchhoff Stress:	$\mathbf{P}$	$= (\det \mathbf{F})\ \mathbf{T}\ \mathbf{F}^{-T}$ $= \mathbf{F}\ \mathbf{S}$
Second Piola-Kirchhoff Stress:	$\mathbf{S}$	$= (\det \mathbf{F})\ \mathbf{F}^{-1}\ \mathbf{T}\ \mathbf{F}^{-T}$ $= \mathbf{F}^{-1}\ \mathbf{P}$

4.4 Power

A fundamental property of a body in motion subjected to forces is *power*, the work per unit time developed by the forces acting on the body. Work is “force times distance” and power is “force times velocity”. In continuum mechanics, the work per unit time – the power – is the function $\mathcal{P} = \mathcal{P}(t)$ defined by

$$\boxed{\mathcal{P} = \int_{\Omega_t} \mathbf{f} \cdot \mathbf{v}\ dx + \int_{\partial\Omega_t} \boldsymbol{\sigma}(\mathbf{n}) \cdot \mathbf{v}\ dA}$$

where $\mathbf{v}$ is the velocity. Since

$$\int_{\partial\Omega_t} \boldsymbol{\sigma}(\mathbf{n}) \cdot \mathbf{v}\ dA = \int_{\partial\Omega_t} \mathbf{Tn} \cdot \mathbf{v}\ dA = \int_{\Omega_t} (\mathbf{v} \cdot \operatorname{div} \mathbf{T} + \mathbf{T} : \operatorname{grad} \mathbf{v})\ dx$$

we have

$$\begin{aligned} \mathcal{P} &= \int_{\Omega_t} \mathbf{v} \cdot (\operatorname{div} \mathbf{Tf})\ dx + \int_{\Omega_t} \mathbf{T} : \mathbf{D}\ dx = \int_{\Omega_t} \mathbf{v} \cdot \varrho \frac{d\mathbf{v}}{dt}\ dx + \int_{\Omega_t} \mathbf{T} : \mathbf{D}\ dx \\ &= \frac{d}{dt}\left(\frac{1}{2}\int_{\Omega_t} \varrho \mathbf{v} \cdot \mathbf{v}\ dx\right) + \int_{\Omega_t} \mathbf{T} : \mathbf{D}\ dx = \frac{d\kappa}{dt} + \int_{\Omega_t} \mathbf{T} : \mathbf{D}\ dx \end{aligned}$$

where κ is the kinetic energy

$$\kappa = \frac{1}{2}\int_{\Omega_t} \varrho \mathbf{v} \cdot \mathbf{v}\ dx$$

and

$$\mathbf{D} = (\operatorname{grad} \mathbf{v})_{sym} = \frac{1}{2}(\operatorname{grad} \mathbf{v} + \operatorname{grad} \mathbf{v}^T)$$

Note that $\mathbf{T} : \operatorname{grad} \mathbf{v} = \mathbf{T} : (\mathbf{D} + \mathbf{W}) = \mathbf{T} : \mathbf{D}$. The quantity $\mathbf{T} : \mathbf{D}$ is called the **stress power**.

In summary, the total power of a body $\mathcal{B}$ in motion is the sum of the time-rate-of change of the kinetic energy and the total stress power:

$$\boxed{\mathcal{P} = \frac{d\kappa}{dt} + \int_{\Omega_t} \mathbf{T} : \mathbf{D}\; dx}$$

Equivalently,

$$\boxed{\int_{\Omega_t} \mathbf{f} \cdot \mathbf{v}\; dx + \int_{\partial\Omega_t} \mathbf{T}\mathbf{n} \cdot \mathbf{v}\; dA = \frac{d\kappa}{dt} + \int_{\Omega_t} \mathbf{T} : \mathbf{D}\; dx}$$

In terms of quantities referred to the reference configuration,

$$\int_{\Omega_0} \mathbf{f}_0 \cdot \dot{\mathbf{u}}\; dX + \int_{\partial\Omega_0} \mathbf{P}\mathbf{n}_0 \cdot \dot{\mathbf{u}}\; dA_0 = \int_{\Omega_0} \mathbf{P} : \dot{\mathbf{F}}\; dX + \frac{d}{dt}\frac{1}{2}\int_{\Omega_0} \varrho_0 \dot{\mathbf{u}} \cdot \dot{\mathbf{u}}\; dX$$

In terms of the second Piola-Kirchhoff stress tensor, the stress power is

$$\int_{\Omega_t} \mathbf{T} : \mathbf{D}\; dx = \int_{\Omega_0} \mathbf{T} : \mathrm{grad}\; \mathbf{v} \det \mathbf{F}\; dX$$

But

$$\begin{aligned}
\mathbf{T} : \mathrm{grad}\; \mathbf{v} \det \mathbf{F} &= T_{ij} \frac{\partial}{\partial X_k}\left(\frac{\partial \varphi_i}{\partial t}\right) \cdot \frac{\partial X_k}{\partial x_j} \det \mathbf{F} \\
&= T_{ij} \dot{F}_{ik} F_{kj}^{-1} \det \mathbf{F} \\
&= \dot{\mathbf{F}}_{ik} (\mathbf{T}_{ij} \det \mathbf{F} F_{kj}^{-1}) \\
&= \dot{F}_{ik} P_{ik} \\
&= \mathbf{F}^T \dot{\mathbf{F}} : \mathbf{S}
\end{aligned}$$

and

$$\mathbf{F}^T \dot{\mathbf{F}} : \mathbf{S} = \left[\frac{1}{2}(\mathbf{F}^T \dot{\mathbf{F}} + \dot{\mathbf{F}}^T \mathbf{F}) + \frac{1}{2}(\mathbf{F}^T \dot{\mathbf{F}} - \dot{\mathbf{F}}^T \mathbf{F})\right] \mathbf{S} = \dot{\mathbf{E}} : \mathbf{S} + \mathbf{0} = \mathbf{S} : \dot{\mathbf{E}}$$

Thus

$$\int_{\Omega_t} \mathbf{T} : \mathbf{D}\; dx = \int_{\Omega_0} \mathbf{S} : \dot{\mathbf{E}}\; dX$$

and, finally,

$$\boxed{\int_{\Omega_0} \mathbf{f}_0 \cdot \dot{\mathbf{u}}\; dX + \int_{\partial\Omega_0} \mathbf{F}\,\mathbf{S}\,\mathbf{n}_0 \cdot \dot{\mathbf{u}}\; dA = \int_{\Omega_0} \mathbf{S} : \dot{\mathbf{E}}\; dX + \frac{d}{dt}\left(\frac{1}{2}\int_{\Omega_0} \varrho_0 \dot{\mathbf{u}} \cdot \dot{\mathbf{u}}\; dX\right)}$$

Chapter 5

The Principle of Conservation of Energy

Energy is a quality of a physical system (e.g. a deformable body) measuring its capacity to do work: a change in energy causes work to be done by the forces acting on the system. A change in energy in time produces a rate of work – i.e. power. So the rate of change of the total energy of a body due to "mechanical processes" (those without a change in temperature) is equal to the mechanical power developed by the forces on the body due to its motion. The total energy is the sum of the kinetic energy (the energy due to motion) κ and the internal energy $\mathcal{E}_{int}$ due to deformation:

$$\text{Total energy } = \kappa + \mathcal{E}_{int}$$

The internal energy depends on the deformation, temperature gradient, and other physical entities. The precise form of this dependency varies from material to material and depends upon the physical "constitution" of the body. In continuum mechanics, it is assumed that a *specific internal energy density* (energy density per unit mass) exists so that

$$\mathcal{E}_{int} = \int_{\Omega_0} \varrho_0 e_0(\mathbf{X}, t)\ dX = \int_{\Omega_t} \varrho e(\boldsymbol{x}, t)\ dx$$

Recall that the kinetic energy is given by

$$\kappa = \frac{1}{2} \int_{\Omega_0} \varrho_0\, \dot{\mathbf{u}} \cdot \dot{\mathbf{u}}\ dX = \frac{1}{2} \int_{\Omega_t} \varrho \mathbf{v} \cdot \mathbf{v}\ dx$$

and that the power $\mathcal{P}$ is

$$\mathcal{P} = \frac{d\kappa}{dt} + \int_{\Omega_t} \mathbf{T} : \mathbf{D}\ dx = \frac{d\kappa}{dt} + \int_{\Omega_0} \mathbf{F}\, \underbrace{\mathbf{S} : \dot{\mathbf{F}}}_{=\ \mathbf{S}:\dot{\mathbf{E}}}\ dX$$

The change in unit time of the total energy (the time-rate-of-change of $\kappa + \mathcal{E}_{int}$) produces power and *heating* of the body. The heating per unit time Q is of the form,

$$Q = \int_{\partial\Omega_t} -\mathbf{q} \cdot \mathbf{n}\ dA + \int_{\Omega_t} r\ dx = \int_{\partial\Omega_0} -\mathbf{q}_0 \cdot \mathbf{n}_0\ dA_0 + \int_{\Omega_0} r_0\ dx$$

where $\mathbf{q}$ is the heat flux entering the body across the surface $\partial\Omega_t$ (minus $\mathbf{q}$ indicated heat entering and not leaving the body) and r is the heat per unit volume generated by internal sources (e.g. chemical reactions) and $\mathbf{q}_0$ and r_0 are their counterparts referred to the reference configurations

$$\mathbf{q}\ \det\mathbf{F}\ \mathbf{F}^{-T} = \mathbf{q}\ \mathrm{Cof}\ \mathbf{F} = \mathbf{q}_0$$

$$r\ \det\mathbf{F} = r_0$$

The principle of conservation of energy asserts that the time-rate-of-change of the total energy is balanced by (equals) the power plus the heating of the body:

$$\boxed{\frac{d}{dt}(\kappa + \mathcal{E}_{int}) = \mathcal{P} + Q}$$

5.1 Local Forms of the Principle of Conservation of Energy

Recalling the definitions of κ, $\mathcal{E}_{int}$, $\mathcal{P}$, and Q we have

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\Omega_t}\varrho\mathbf{v}\cdot\mathbf{v}\ dx + \int_{\Omega_t}\varrho e\ dx\right) = \int_{\Omega_t}\mathbf{T}:\mathbf{D}\ dx + \frac{d\kappa}{dt} - \int_{\partial\Omega_t}\mathbf{q}\cdot\mathbf{n}\ dA + \int_{\Omega_t} r\ dx$$

Thus

$$\int_{\Omega_t}\left(\varrho\frac{de}{dt} - \mathbf{T}:\mathbf{D} + \mathrm{div}\ \mathbf{q} - r\right)dx = 0$$

or

$$\boxed{\varrho\frac{de}{dt} = \mathbf{T}:\mathbf{D} - \mathrm{div}\ \mathbf{q} + r}$$

For equivalent results referred to the reference configuration Ω_0, we have

$$\frac{d}{dt}\left(\kappa + \int_{\Omega_0}\varrho_0 e_0\ dX\right) = \int_{\Omega_0}\mathbf{S}:\dot{\mathbf{E}}\ dX + \frac{d\kappa}{dt} - \int_{\partial\Omega_0}\mathbf{q}_0\cdot\mathbf{n}_0\ dA_0 + \int_{\Omega_0} r_0\ dX$$

or

$$\int_{\Omega_0}\left(\varrho_0\dot{e}_0 - \mathbf{S}:\dot{\mathbf{E}} + \mathrm{Div}\ \mathbf{q}_0 - r_0\right)dX = 0$$

Thus, locally,

$$\boxed{\varrho_0\dot{e}_0 = \mathbf{S}:\dot{\mathbf{E}} - \mathrm{Div}\ \mathbf{q}_0 + r_0}$$

Chapter 6

Thermodynamics of Continua and the Second Law

In contemplating the thermal and mechanical behavior of the physical universe, it is convenient to think of thermomechanical systems as some open region $\mathcal{S}$ of three-dimensional Euclidean space containing, perhaps, one or more deformable bodies. Such a system is *closed* if it does not exchange matter with the complement of $\mathcal{S}$, called the *exterior* of $\mathcal{S}$. There can be an exchange of energy between $\mathcal{S}$ and its exterior due to work of external forces and heating (cooling) of $\mathcal{S}$ by the transfer of heat from $\mathcal{S}$ to its exterior. A system is a *thermodynamic system* if the only exchange of energy with its exterior is a possible exchange of heat and of work done by body and contact forces acting on $\mathcal{S}$. The *thermodynamic state* of a system $\mathcal{S}$ is characterized by the values of so-called *thermodynamic state variables*, such as temperature, mass density, etc. which reflect the mechanical and thermal condition of the system; we may write $\mathcal{T}(\mathcal{S}, t)$ for the thermodynamic state of system $\mathcal{S}$ at time t. If the thermodynamic system does not evolve in time, it is in *thermodynamic equilibrium*. The transition from one state to another is a *thermodynamic process*.

Thus, we may think of the thermodynamic state of a system as the values of certain fields that provide all of the information needed to characterize the system: stress, strain, velocity, etc., and quantities that measure the hotness or coldness of the system and possible rates of change of these quantities. The *absolute temperature* $\theta \in \mathbb{R}$, $\theta > 0$ provides a measure of the hotness of a system and a characterization of the thermal state of a system. Two closed systems $\mathcal{S}_1$ and $\mathcal{S}_2$ are in thermal equilibrium with each other (and with a third system $\mathcal{S}_3$) if they share the same value of θ. Thus, θ is a state variable. The second quantity needed to define the thermodynamic state of a system is the *entropy*, which represents a bound on the amount of heating a system can receive at a given temperature θ.

In continuum mechanics, the total entropy H of a body is the integral of a specific entropy density η (entropy density per unit mass):

$$H = \int_{\Omega_t} \varrho\eta(\boldsymbol{x}, t)\ dx$$

In classical thermodynamics, the change in entropy between two states $\mathcal{T}(\mathcal{S}, t_1)$ and $\mathcal{T}(\mathcal{S}, t_2)$ of a system measures the quantity of heat received per unit temperature. When $\theta =$ constant, the classical condition is

$$H(\mathcal{S}, t_1) - H(\mathcal{S}, t_2) - \frac{Q}{\theta} = 0$$

for reversible processes, and

$$H(\mathcal{S},t_1) - H(\mathcal{S},t_2) - \frac{Q}{\theta} \geq 0$$

for all possible processes, $H(\mathcal{S},t)$ being the total entropy of system $\mathcal{S}$ at time t and Q being the heating.

In continuum mechanics, we analogously require

$$\boxed{\frac{dH}{dt} + \int_{\partial\Omega_t} \frac{1}{\theta}\mathbf{q}\cdot\mathbf{n}\, dA - \int_{\Omega_t} \frac{r}{\theta}\, dx \geq 0}$$

This is called the *Second Law of Thermodynamics*: the total entropy production per unit time is always ≥ 0. Locally, we have

$$\int_{\Omega_t} \left(\varrho\frac{d\eta}{dt} + \mathrm{div}\ \frac{\mathbf{q}}{\theta} - \frac{r}{\theta} \right) dx \geq 0$$

or

$$\boxed{\varrho\frac{d\eta}{dt} + \mathrm{div}\ \frac{\mathbf{q}}{\theta} - \frac{r}{\theta} \geq 0}$$

In the material description,

$$\boxed{\varrho_0\dot{\eta}_0 + \mathrm{Div}\ \frac{\mathbf{q}_0}{\theta} - \frac{r_0}{\theta} \geq 0}$$

This relation is called the *Clausius-Duhem inequality.*

Material (LAGRANGIAN)	**Spatial** (EULERIAN)
Conservation of Mass	
$\varrho_0 = \varrho \det \mathbf{F}$	$\dfrac{\partial \varrho}{\partial t} + \text{div}\ (\varrho \mathbf{v}) = 0$
Conservation of Linear Momentum	
$\text{Div}\ \mathbf{FS} + \mathbf{f}_0 = \varrho_0 \partial^2 \mathbf{u} / \partial t^2$	$\text{div}\ \mathbf{T} + \mathbf{f} = \varrho \left(\dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \text{grad}\ \mathbf{v} \right)$
Conservation of Angular Momentum	
$\mathbf{S} = \mathbf{S}^T$	$\mathbf{T} = \mathbf{T}^T$
Conservation of Energy	
$\varrho_0 \dot{e}_0 = \mathbf{S} : \dot{\mathbf{E}} - \text{Div}\ \mathbf{q}_0 + r_0$	$\varrho \dfrac{\partial e}{\partial t} + \mathbf{v} \cdot \text{grad}\ e = \mathbf{T} : \mathbf{D} - \text{div}\ \mathbf{q} + r$
Second Law of Thermodynamics (The Clausius-Duhem Inequality)	
$\varrho_0 \dot{\eta}_0 + \text{Div}\ \dfrac{\mathbf{q}_0}{\theta} - \dfrac{r_0}{\theta} \geq 0$	$\varrho \dfrac{\partial \eta}{\partial t} + \mathbf{v} \cdot \text{grad}\ \eta + \text{div}\ \dfrac{\mathbf{q}}{\theta} - \dfrac{r}{\theta} \geq 0$

Figure 6.1: Summary.

Chapter 7

Constitutive Equations

Given initial data (i.e. given all of the information needed to describe the body $\mathcal{B}$ and its thermodynamic state $\mathcal{S}$ while it occupies its reference configuration at time $t = 0$),

$$(\Omega_0,\ \partial\Omega_0,\ \mathbf{f}_0(\mathbf{X}, t),\ t \in [0, T],\ \mathbf{v}_0 = \dot{\mathbf{u}}(\mathbf{X}, 0),\ \theta_0(\mathbf{X}) = \theta(\mathbf{X}, 0), \ldots),$$

we wish to use the equations of continuum mechanics (conservation of mass, energy, balance of linear and angular momentum, the second law of thermodynamics) to determine the behavior of the body (the motion, deformation, temperature, stress, heat flux, entropy, . . .) at each $\mathbf{X} \in \bar{\Omega}_0$ for any time t in some interval $[0, T]$.

Unfortunately, we do not have enough information to solve this problem. The balance laws apply to all materials, and we know that different materials respond to the same stimuli in different ways depending on their *constitution*. To complete the problem (to "close" the system of equations), we must supplement the basic equations with *constitutive equations* that characterize the material(s) of which the body is composed.

In recognizing that additional relations are needed to close the system, we ask what variable can we identify as natural "dependent variables" and which are "independent". Our choice is to choose as primitive (state) variables those features of the response naturally experienced by observation – by our physical senses: the motion (or displacement), the rate of motion, the deformation or rate of deformation, the hotness or coldness (the temperature) or its gradients, etc., or the "histories" of these quantities. If we then knew how, for example, the stress, heat flux, internal energy, and entropy were dependent on the state variables *for a given material*, we would hope to have sufficient information to completely characterize the behavior of the body under the action of given stimuli (loads, heating, etc.). Thus, we seek constitutive equations of the type,

$$\boxed{\begin{aligned} \mathbf{T} &= \mathbf{T}(\boldsymbol{x}, t, \wedge) \\ \mathbf{q} &= \mathbf{Q}(\boldsymbol{x}, t, \wedge) \\ e &= \xi(\boldsymbol{x}, t, \wedge) \\ \eta &= \mathcal{H}(\boldsymbol{x}, t, \wedge) \end{aligned}}$$

where, for the moment, $\wedge$ denotes everything we might expect to influence the stress, heat flux, internal energy, and entropy at a point $\boldsymbol{x} \in \Omega_t$ at time t (for example, $\wedge = (\mathbf{F}, \mathbf{C}, \theta, \nabla\theta, \mathbf{L}, \mathbf{D}, \ldots)$). The functions $\mathbf{T}$, $\mathbf{Q}$, $\mathcal{E}$, $\mathcal{H}$ defining the constitutive equations are

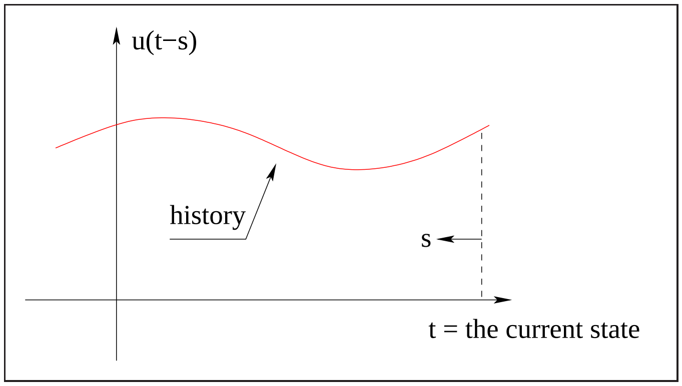

Figure 7.1: Time history.

called *response functions* (or, response functionals).

7.1 Rules and Principles for Constitutive Equations

What rules-of-thumb or even fundamental principles govern the forms of the response functions for real materials? We list some of the most important rules and principles:

1. The Principle of Determinism
The behavior of a material at a point $\mathbf{X}$ occupying $\boldsymbol{x}$ at time t is determined by the history of the primitive variables $\wedge$.

In other words, the conditions that prevail at $(\boldsymbol{x}, t)$ depend upon the past behavior of the array $\wedge$ (not the future). In general, a "history" of a function $u = u(\boldsymbol{x}, t)$, up to current time t, is the set

$$u^t(s) = \{u(\boldsymbol{x}, t - s),\ s \geq 0\}$$

The history from the reference configuration is then $u(\boldsymbol{x}, t-s)$, $t \geq s \geq 0$ (see Fig. 7.1). Then we should, in general, replace $\wedge$ by $\wedge^t(s)$ in the constitutive equations unless conditions prevail (in the constitution of the material) that suggest the response depends only on values at the current time.

2. The Principle of Material Frame Indifference – or Principle of Objectivity
The form of the constitutive equations must be invariant under changes of the frame of reference: they must be independent of the observer.

For example, if we rotate the coordinates $\boldsymbol{x}$ into a system $\boldsymbol{x}^*$ (and translate the origin)

$$\left.\begin{aligned} \boldsymbol{x}^* &= \mathbf{Q}(t)\boldsymbol{x} + \mathbf{c}(t) \\ \mathbf{Q}^T(t) &= \mathbf{Q}^{-1}(t) \end{aligned}\right\} \quad \Rightarrow \text{"Change of the Observer"}$$

and change the clock: $t^* = t - a$, the constitutive functions should remain invariant. For example,

$$\text{if } \mathbf{T} = \boldsymbol{\mathcal{T}}(\boldsymbol{x}, t), \quad \text{then } \mathbf{T}^* = \boldsymbol{\mathcal{T}}(\boldsymbol{x}^*, t^*)$$

where $\boldsymbol{\mathcal{T}}$ is the same function in both equations and

$$\mathbf{T}^* = \mathbf{Q}(t)\mathbf{T}\mathbf{Q}(t)^T$$

3. The Principle of Physical Consistency
The constitutive equations cannot violate or contradict the physical principles of mechanics (the conservation of mass, energy, balance of momentum, or the Clausius-Duhem inequality).

4. The Principle of Material Symmetry
For every material, there is a group $\mathcal{G}$ of unimodular transformations of the material coordinates, called the isotropy group of the material, under which the forms of the constitutive functions remain invariant.

- The unimodular tensors H are those for which $\det H = \pm 1$.
- The group $\mathcal{G}$ is a group of unimodular linear transformations with group operation = composition (matrix multiplication). Thus,
    1. $\mathbf{A}, \mathbf{B} \in \mathcal{G} \Rightarrow \mathbf{AB} \in \mathcal{G}$
    2. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
    3. $\exists\, \mathbf{1} \in \mathcal{G}$ such that $\mathbf{A} \cdot \mathbf{1} = \mathbf{A}$
    4. $\forall\, \mathbf{A} \in \mathcal{G}, \exists \mathbf{A}^{-1} \in \mathcal{G}$, such that $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{1}$
- Let $\mathcal{O}^+$ be the group of rotations (the proper orthogonal group). If, at a material point $\mathbf{X}$, the symmetry group $\mathcal{G} = \mathcal{O}^+$, then the material is *isotropic* at $\bar{\mathbf{X}}$; otherwise, it is *anisotropic*.

5. The Principle of Local Action
The dependent (primitive) constitutive variables at $\mathbf{X}$ are not affected by the actions of independent variables ($(\mathbf{T}, \mathbf{q}, e, \eta)$ for example) at points distant from X.

There are other rules. For example:

Dimensional consistency. Terms in the constitutive equations must, of course, be dimensionally consistent (this can be interpreted as a corollary to Principle 3). Beyond this, dimensional analysis can be used to extract information on the constitutive variables.

Existence, well-posedness. The constitutive equations must be such that there exist solutions to properly-posed boundary and initial-value problems resulting from use of the equations of continuum mechanics.

Equipresence. When beginning to characterize the response functions of a material, be sure that the dependent variables in $\wedge$ are "equally present" — in other words, until evidence from some other source suggests otherwise, assume that all the constitutive functions for $\mathbf{T}$, $\mathbf{q}$, e, η depend on the same full list of variables $\wedge$.

Chapter 8

Examples and Applications

8.1 Principle of Material Frame Indifference

8.1.1 Solids

Consider a "change in the observer":

$$\boldsymbol{x}^*(t) = \mathbf{Q}(t)\boldsymbol{x}(t) + \mathbf{c}(t)$$
$$\mathbf{Q}(t)^{-1} = \mathbf{Q}(t)^T$$

where $\boldsymbol{x}(t) = \boldsymbol{\varphi}(\mathbf{X}, t)$. Clearly $\mathbf{F}^* = \boldsymbol{\nabla}\boldsymbol{x}^*$ and $\mathbf{F} = \boldsymbol{\nabla}\boldsymbol{x}$, so that

$$\boxed{\mathbf{F}^* = \mathbf{QF} \qquad \text{and} \qquad \det \mathbf{F}^* = \det \mathbf{F}}$$

If $\mathbf{n}^* = \mathbf{Qn}$ and $\boldsymbol{\sigma}^*(\mathbf{n}^*) = \mathbf{Q}\boldsymbol{\sigma}(\mathbf{n})$ then

$$\mathbf{T}^*\mathbf{n}^* = \mathbf{QTQ}^T\mathbf{n}^* \quad \Rightarrow \quad \boxed{\mathbf{T}^* = \mathbf{QTQ}^T}$$

Suppose

$$\mathbf{T} = \boldsymbol{\mathcal{T}}(\mathbf{F})$$

$$\text{(Local Action} \Rightarrow \mathbf{T} \text{ depends on} \boldsymbol{\nabla}\boldsymbol{\varphi}, \text{ not } \boldsymbol{\varphi})$$

Then

$$\mathbf{T}^* = \boldsymbol{\mathcal{T}}(\mathbf{F}^*)$$

Thus, the material response function is independent of the observer, if, $\forall\, \mathbf{Q}(t)$, $\mathbf{Q}(t)^T = \mathbf{Q}(t)^{-1}$, we have

$$\boxed{\boldsymbol{\mathcal{T}}(\mathbf{QF}) = \mathbf{Q}\boldsymbol{\mathcal{T}}(\mathbf{F})\mathbf{Q}^T}$$

Application: If $\boldsymbol{\mathcal{T}}(\mathbf{F}) = \mathbf{T}$, then $\boldsymbol{\mathcal{T}}(\mathbf{F}) = \mathbf{R}\boldsymbol{\mathcal{T}}(\mathbf{U})\mathbf{R}^T$, since $\boldsymbol{\mathcal{T}}(\mathbf{R}^T\mathbf{F}) = \mathbf{R}\boldsymbol{\mathcal{T}}(\mathbf{R}^T\mathbf{RU})\mathbf{R}^T = \mathbf{R}\boldsymbol{\mathcal{T}}(\mathbf{U})\mathbf{R}^T$. This shows that we can also express $\mathbf{T}$ as a function of $\mathbf{U}$ or, since $\mathbf{U}^2 = \mathbf{C}$, of $\mathbf{C}$ or $\mathbf{E}$: i.e. $\mathbf{T} = \boldsymbol{\mathcal{T}}(\mathbf{F}) = R\boldsymbol{\mathcal{T}}(\mathbf{U})\mathbf{R}^T = \mathbf{FU}^{-1}\boldsymbol{\mathcal{T}}(\mathbf{C}^{1/2})\mathbf{U}^{-1}\mathbf{F}^T = \mathbf{F}\widehat{\boldsymbol{\mathcal{T}}}(\mathbf{C})\mathbf{F}^T$. □

8.1.2 Fluids

Suppose

$$\mathbf{T} = \boldsymbol{\mathcal{F}}(\mathbf{L}) \qquad \text{(a fluid)}$$

and $\boldsymbol{\mathcal{F}}(\mathbf{0}) = -p\,\mathbf{I}$, $p =$ pressure (when there is no motion, we want the stress field to be a "hydrostatic" pressure, $p = p(\boldsymbol{x}, t)$). Suppose also that

$$\text{tr } \mathbf{L} = \text{div } \mathbf{v} = 0$$

Then

$$\mathbf{T} = -p\,\mathbf{I} + \boldsymbol{\mathcal{F}}_0(\mathbf{D})$$

$\boldsymbol{\mathcal{F}}_0(\mathbf{0}) = \mathbf{0}$. If $\boldsymbol{\mathcal{F}}_0$ is linear in $\mathbf{D}$, then a necessary and sufficient condition that $\boldsymbol{\mathcal{F}}_0(\mathbf{D})$ is invariant under a change of the observer is that there exists a constant (or scalar) $\mu = \mu(t) > 0$, such that $\boldsymbol{\mathcal{F}}_0(\mathbf{D}) = 2\mu\mathbf{D}$. Then

$$\boxed{\mathbf{T} = -p\,\mathbf{I} + 2\mu\mathbf{D}}$$

This is the classical constitutive equation for Cauchy stress in a Newtonian fluid (a viscous incompressible fluid) where μ is the *viscosity of the fluid.*

8.2 The Navier-Stokes Equations for Incompressible Flow

Conservation of Mass

$$\frac{\partial \varrho}{\partial t} + \text{div } (\varrho \mathbf{v}) = 0$$

For an incompressible fluid, we have

$$0 = \frac{d}{dt}\int_{\Omega_t} dx = \int_{\Omega_0} \dot{\overline{\det \mathbf{F}}}\; dX = \int_{\Omega_0} \det \mathbf{F} \text{ div } \mathbf{v}\; dX = \int_{\Omega_t} \text{div } \mathbf{v}\; dx$$

which implies that div $\mathbf{v} = 0$. Thus

$$0 = \frac{\partial \varrho}{\partial t} + \mathbf{v}\cdot \text{grad } \varrho + \varrho \text{div } \mathbf{v} = \frac{\partial \varrho}{\partial t} + \mathbf{v}\cdot\text{grad } \varrho = \frac{d\varrho}{dt} \qquad \Rightarrow \varrho \text{ is constant.}$$

Conservation of Momentum

$$\varrho\frac{\partial \mathbf{v}}{\partial t} + \varrho\,\mathbf{v}\cdot \text{grad } \mathbf{v} - \text{div } \mathbf{T} = \mathbf{f}, \qquad \forall(\boldsymbol{x}, t) \in \Omega_t \times (0, T)$$

$$\mathbf{T} = \mathbf{T}^T$$

Constitutive Equation

$$\begin{aligned}\mathbf{T} &= -p\,\mathbf{I} + 2\mu\mathbf{D} \qquad (p = \pi) \\ \mathbf{D} &= \frac{1}{2}\left(\text{grad } \mathbf{v} + (\text{grad } \mathbf{v})^T\right)\end{aligned}$$

The Navier-Stokes Equations

$$\boxed{\begin{aligned}\varrho\frac{\partial \mathbf{v}}{\partial t} + \varrho\,\mathbf{v}\cdot\text{grad } \mathbf{v} - \mu\Delta\mathbf{v} + \text{grad } p &= \mathbf{f} \\ \text{div } \mathbf{v} &= 0\end{aligned}}$$

where Δ = vector Laplacian = div grad.

Application: An Initial-Boundary-Value Problem

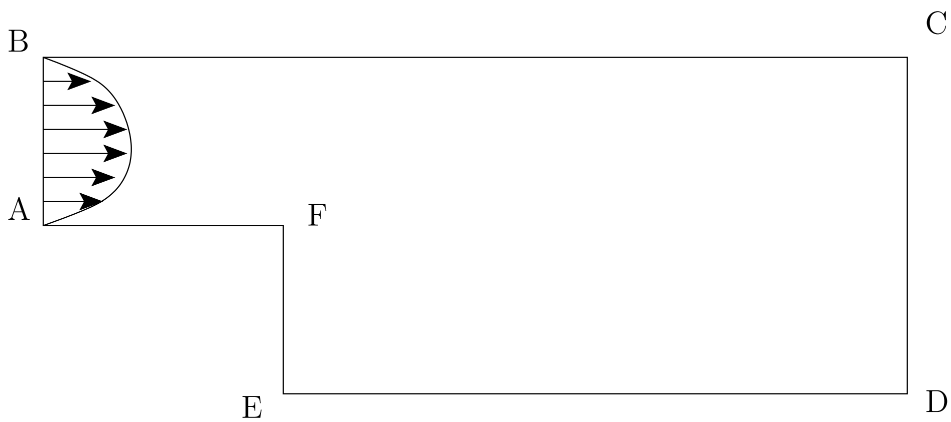

Figure 8.1: Geometry of the backstep channel flow.

Initial conditions:

$$\mathbf{v}(\boldsymbol{x}, 0) = \mathbf{v}_0(\boldsymbol{x})$$

where the initial field must satisfy div $\mathbf{v}_0 = 0$, which implies that

$$\int_{\partial\Omega_0} \mathbf{v}_0 \cdot \mathbf{n}\, dA = 0, \qquad \text{i.e. } \mathbf{v}_0 \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_0.$$

Boundary conditions:

1. On segment BC and DE ∪ EF ∪ FA, we have the no-slip boundary condition $\mathbf{v} = 0$.

2. On the "in-flow boundary" AB, we prescribe the Poiseuille flow velocity profile (see Fig. 8.2):

$$\begin{cases} v_1(0, x_2, t) = \left(1 - \left(\dfrac{2x_2}{a}\right)^2\right) U_0 \\ v_2(0, x_2, t) = 0 \end{cases}$$

3. On the "out-flow boundary" CD, there are several possibilities. A commonly used one is

$$\begin{cases} -p + \mu \left.\dfrac{\partial v_1}{\partial x_1}\right|_{x_1=L} = \mathbf{T}\mathbf{e}_1|_{x_1=L} = 0 \\ v_2(L, x_2, t) = 0 \end{cases}$$

where L is the length of the channel, i.e. $L = |\mathrm{BC}|$.

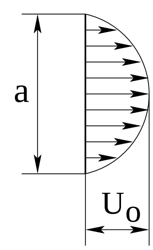

Figure 8.2: Poiseuille flow velocity profile.

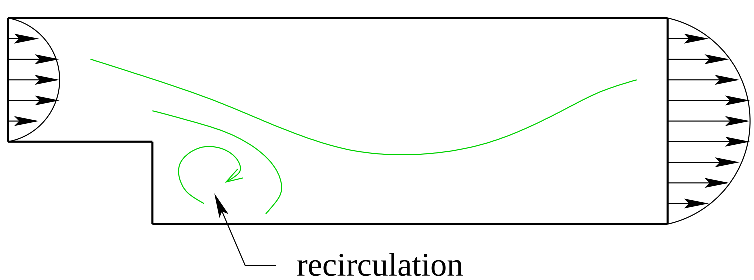

Figure 8.3: Recirculation in backstep channel flow.

8.3 Application of the Principle of Physical Consistency

Instead of writing the equations governing the conservation of energy or the Clausius-Duhem inequality in terms of the internal energy e per unit volume, it is often convenient to replace e by the Helmholtz free energy

$$\psi = e - \eta\,\theta$$

Then the Clausius-Duhem inequality becomes (e.g.)

$$-\varrho_0\dot{\psi}_0 - \varrho_0\eta_0\dot{\theta} + \mathbf{S} : \dot{\mathbf{E}} - \frac{1}{\theta}\,\mathbf{q}_0 \cdot \nabla\theta \geq 0 \qquad \text{(See Exercise)}$$

Suppose we have a constitutive equation for ψ_0 which we initially take to be of the form

$$\psi_0 = \Psi(\mathbf{E}, \theta, \nabla\theta)$$

Then

$$\dot{\psi}_0 = \frac{\partial\Psi}{\partial\mathbf{E}} : \dot{\mathbf{E}} + \frac{\partial\Psi}{\partial\theta}\dot{\theta} + \frac{\partial\Psi}{\partial\nabla\theta} : \nabla\dot{\theta}$$

and

$$-\varrho_0\left(\eta_0 + \frac{\partial\Psi}{\partial\theta}\right)\dot{\theta} + \left(\mathbf{S} - \varrho_0\frac{\partial\Psi}{\partial\mathbf{E}}\right) : \dot{\mathbf{E}} + \varrho_0\frac{\partial\Psi}{\partial\nabla\theta} : \nabla\dot{\theta} - \frac{1}{\theta}\mathbf{q}_0 \cdot \nabla\theta \geq 0$$

This implies that

$$\boxed{\eta_0 = -\frac{\partial\Psi}{\partial\theta}, \quad \mathbf{S} = \varrho_0\frac{\partial\Psi}{\partial\mathbf{E}}, \quad \frac{\partial\Psi}{\partial\nabla\theta} = 0}$$

Thus, the Clausius-Duhem inequality allows us to 1) conclude that Ψ does not depend on $\nabla\theta$, and 2) conclude that η_0 and $\mathbf{S}$ can be characterized through a single energy functional Ψ.

Remark: The quantity

$$\delta_0 = \mathbf{S} : \dot{\mathbf{E}} - \varrho_0(\dot{\psi}_0 + \eta_0\dot{\theta}) = \text{Div } \mathbf{q}_0 - r_0 + \varrho_0\theta\dot{\eta}_0$$

is called the *internal dissipation.* According to the second law (Clausius-Duhem)

$$\delta_0 - \frac{1}{\theta}q_0 \cdot \nabla\theta \geq 0$$

This is the Clausius-Planck inequality. If $\delta_0 = 0$, it asserts that heat must flow from hot to cold. But δ_0 may not be zero!

8.4 Heat Conduction

Ignoring motion and deformation for the moment, consider a rigid body being heated by some outside source. The constitutive equations are

$$\psi_0 = \Psi(\theta) = \frac{1}{2}c\,\theta^2 \quad \text{(c constant)}$$
$$\mathbf{q}_0 = k\nabla\theta \quad \text{(Fourier's Law)}$$

Then:

$$\eta_0 = -\frac{\partial \Psi}{\partial \theta} = -c\,\theta, \qquad \text{and} \quad \mathbf{S} = \varrho_0 \frac{\partial \Psi}{\partial \mathbf{E}} = 0 \quad \text{(irrelevant)}$$

In a reversible process, $\delta_0 = 0$. So

$$\text{Div } \mathbf{q}_0 - r_0 + \theta \varrho_0 \dot{\eta}_0 = 0$$

Setting $\theta \dot{\eta}_0 = -c\,\theta \dot{\theta} \approx -c\,\theta_0 \dot{\theta}$, where θ_0 = reference temperature > 0, we get

$$\varrho_0 c_0 \frac{\partial \theta}{\partial t} - \nabla \cdot k \nabla \theta = -r_0$$

where $c_0 = c\,\theta_0$. This is the classical heat conduction (diffusion) equation.

8.5 Theory of Elasticity

We consider a deformable body $\mathcal{B}$ under the action of forces (body forces $\mathbf{f}$ and prescribed contact forces $\boldsymbol{\sigma}(\mathbf{n}, \boldsymbol{x}) = \mathbf{g}(\boldsymbol{x})$ on $\partial\Omega_0$). The body is constructed on a material which is homogeneous and isotropic and is subjected to only isothermal (θ = const.) and adiabatic ($\mathbf{q} = \mathbf{0}$) processes. The sole constitutive equation is

$$\psi = \text{free energy} = \widehat{\Psi}(\mathbf{X}, t, \theta, \mathbf{E}) = \Psi(\mathbf{E})$$

The constitutive equation for stress is thus

$$\mathbf{S} = \left. \frac{\partial \Psi(\mathbf{E})}{\partial \mathbf{E}} \right|_{sym}$$

In this case, the free energy is called the *stored energy function*, or the *strain energy function*. Since $\Psi()$ (and $\partial\psi/\partial\mathbf{E}$) must be form–invariant under changes of the observer, and since $\mathcal{B}$ is isotropic, $\Psi(\mathbf{E})$ must depend on invariants of $\mathbf{E}$:

$$\Psi(\mathbf{E}) = W(I_E, II_E, III_E)$$

Or, since $\mathbf{E} = (\mathbf{C} - I)/2$, we could also write Ψ as a function of invariants of $\mathbf{C}$

$$\Psi = \widehat{W}(I_C, II_C, III_C)$$

The constitutive equation for stress is then

$$\boxed{\begin{aligned} \mathbf{S} &= \frac{\partial W}{\partial I_E} \cdot \frac{\partial I_E}{\partial \mathbf{E}} + \frac{\partial W}{\partial II_E} \cdot \frac{\partial II_E}{\partial \mathbf{E}} + \frac{\partial W}{\partial III_E} \cdot \frac{\partial III_E}{\partial \mathbf{E}} \\ &= \frac{\partial \widehat{W}}{\partial I_C} \cdot \frac{\partial I_C}{\partial \mathbf{C}} + \frac{\partial \widehat{W}}{\partial II_C} \cdot \frac{\partial II_C}{\partial \mathbf{C}} + \frac{\partial \widehat{W}}{\partial III_C} \cdot \frac{\partial III_C}{\partial \mathbf{C}} \end{aligned}}$$

and we note that

$$\boxed{\begin{aligned} \frac{\partial I_E}{\partial \mathbf{E}} &= \mathbf{I} \\ \frac{\partial I\!I_E}{\partial \mathbf{E}} &= (\text{tr } \mathbf{E}^{-1})\mathbf{I} - \mathbf{E}^{-T}\text{Cof } \mathbf{E} \\ \frac{\partial I\!I\!I_E}{\partial \mathbf{E}} &= \text{Cof } \mathbf{E} \end{aligned}}$$

Materials for which the stress is derivable from a stored energy potential are called *hyperelastic* materials.

The governing equations are

$$\text{Div}\left[(\mathbf{I} + \boldsymbol{\nabla}\mathbf{u})\left.\frac{\partial W}{\partial \mathbf{E}}\right|_{sym}\right] + \mathbf{f}_0 = \varrho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2}$$

with

$$\begin{aligned} \mathbf{F} &= (\mathbf{I} + \boldsymbol{\nabla}\mathbf{u}) \\ \mathbf{S} &= \left.\frac{\partial W}{\partial \mathbf{E}}\right|_{sym} = \mathcal{T}(I_{\mathbf{E}}, I\!I_{\mathbf{E}}, I\!I\!I_{\mathbf{E}}, \mathbf{E}) \\ \mathbf{E} &= \frac{1}{2}(\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}^T + \boldsymbol{\nabla}\mathbf{u}^T\boldsymbol{\nabla}\mathbf{u}) \end{aligned}$$

Linear Elasticity

$$\mathbf{E} \approx \mathbf{e} = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}^T)$$

$$W = \frac{1}{2}E^{ijk\ell}e_{k\ell}\, e_{ij}$$

$$S_{ij} = \frac{\partial W}{\partial e_{ij}} = E^{ijk\ell}e_{k\ell} = E^{ijk\ell}\frac{\partial u_k}{\partial X_\ell} \quad \text{(Hooke's Law)}$$

$$E_{ijk\ell} = E_{jik\ell} = E_{ij\ell k} = E_{k\ell ij}$$

Then, for small strains/displacements,

$$\frac{\partial}{\partial X_j}\left(E_{ijk\ell}\frac{\partial u_k}{\partial X_\ell}\right) + f_{0i} = \varrho\frac{\partial^2 u_i}{\partial t^2}$$

For isotropic materials,

$$E_{ijk\ell} = \lambda\delta_{ij}\delta_{rs} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$$

where λ, μ are the Lamé constants:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}$$

E is the Young's Modulus and ν the Poisson's Ratio.

Then

$$\begin{aligned} \mathbf{S} &= \lambda\ (\mathrm{tr}\ \mathbf{E})\mathbf{I} + 2\mu\mathbf{E} = \lambda\ \mathrm{div}\ \mathbf{u}\ \mathbf{I} + 2\mu(\boldsymbol{\nabla}\mathbf{u})_{sym} \\ S_{ij} &= \lambda\ \delta_{ij}\left(\frac{\partial u_k}{\partial X_k}\right) + \mu\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \end{aligned}$$

and the Lamé equations of elastostatics ($\partial^2\mathbf{u}/\partial t^2 = \mathbf{0}$) are

$$\lambda\frac{\partial^2 u_k}{\partial X_j \partial X_k} + \mu\left(\frac{\partial^2 u_i}{\partial X_j \partial X_j} + \frac{\partial^2 u_i}{\partial X_j \partial X_i}\right) = f_{0j}, \quad 1 \le i,j \le 3$$

8.6 Exercises

1. It is often considered useful to write the first and second laws of thermodynamics in terms of the so-called free energy rather than the internal energy. The scalar field,

$$\psi = e - \theta\eta$$

is called the *Helmholtz free energy* (per unit volume). Show that

$$\frac{d\psi}{dt} = \mathbf{T} : \mathbf{D} - \text{div } \mathbf{q} + r - \theta\frac{d\eta}{dt} - \eta\frac{d\theta}{dt}$$

and that

$$-\frac{d\psi}{dt} - \eta\frac{d\theta}{dt} + \mathbf{T} : \mathbf{D} - \frac{1}{\theta}\mathbf{q} \cdot \nabla\theta \geq 0$$

or, equivalently,

$$-\dot{\psi}_0 - \eta_0\dot{\theta} + \mathbf{S} : \dot{\mathbf{E}} - \frac{1}{\theta}\mathbf{q}_0 \cdot \nabla\theta \geq 0$$

2. Consider the small deformations and heating of a thermo-elastic solid constructed of a material characterized by the following constitutive equations:

Free energy: $\psi_0 = \frac{1}{2}\lambda(\text{tr } \mathbf{e})^2 + \mu\mathbf{e} : \mathbf{e} + c(\text{tr } \mathbf{e})\theta + \frac{c_0}{2}\theta^2$

Heat Flux: $\mathbf{q}_0 = k\nabla\theta$

where

$\mathbf{e} = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}^T) =$ the "infinitesimal" strain tensor ($\approx \mathbf{E}$)
$\mathbf{u} =$ the displacement field
$\theta =$ the temperature field
$\lambda, \mu, c, c_0, k =$ material constants

A body $\mathcal{B}$ is constructed of such a material and is subjected to body forces $\mathbf{f}_0$ and to surface contact forces $\mathbf{g}$ on a portion Γ_g of its boundary $\Gamma_g \subset \partial\Omega_0$. On the remainder of its boundary, $\Gamma_u = \partial\Omega_0 \backslash \Gamma_g$, the displacements $\mathbf{u}$ are prescribed as zero ($\mathbf{u} = \mathbf{0}$ on Γ_u). The mass density of the body is ϱ_0, and, when in its reference configuration at time $t = 0$, $\mathbf{u}(\boldsymbol{x}, 0) = \mathbf{u}_0(\boldsymbol{x})$, $\partial\mathbf{u}(\boldsymbol{x}, 0)/\partial t = \mathbf{v}_0(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega_0$, where $\mathbf{u}_0$ and $\mathbf{v}_0$ are given functions. A portion Γ_q of the boundary is heated, resulting in a prescribed heat flux $h = \mathbf{q} \cdot \mathbf{n}$, and the complementary boundary, $\Gamma_\theta = \partial\Omega_0 \backslash \Gamma_q$ is subjected to a prescribed temperature $\theta(\boldsymbol{x}, t) = \tau(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \Gamma_\theta$ (e.g. is immersed in ice water).

Develop a mathematical model of this physical phenomena (a set of partial differential equations, boundary and initial conditions): the dynamic, thermomechanical behavior of a thermoelastic solid.

Chapter 9

Assignments

Things you should know:

- Linear algebra and matrix theory
- Vector calculus
- Index notation
- Introductory real analysis

Index Notation and Symbolic Notation

- Let $\mathbf{e}_i$, $i = 1, 2, 3$ be an orthonormal basis, i.e.

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

- Let $\mathbf{a}$ be a vector: $\mathbf{a} = a_i\, \mathbf{e}_i$ (repeated indices are summed). Two vectors $\mathbf{a}$ and $\mathbf{b}$ are equal, i.e. $\mathbf{a} = \mathbf{b}$ if $a_i = b_i$, $i = 1, 2, 3$.
- Cross product: $\mathbf{a} \times \mathbf{b} = \mathcal{E}_{ijk}\, a_i\, b_j\, \mathbf{e}_k$

$$\mathcal{E}_{ijk} = \begin{cases} 1, & \text{if } ijk = \text{even permutation} \\ -1, & \text{if } ijk = \text{odd permutation} \\ 0, & \text{if } ijk \text{ is not a permutation} \end{cases}$$

- Nabla: $\nabla = \mathbf{e}_k \dfrac{\partial}{\partial x_k}$
- Divergence of a vector (denoted div $\mathbf{v}$ or $\nabla \cdot \mathbf{v}$):

$$\nabla \cdot \mathbf{v} = \mathbf{e}_k \frac{\partial}{\partial x_k} \cdot v_j \mathbf{e}_j = \frac{\partial v_j}{\partial x_k} \mathbf{e}_k \cdot \mathbf{e}_j = \frac{\partial v_j}{\partial x_k} \delta_{kj} = \frac{\partial v_k}{\partial x_k} (= v_{k,\,k} \text{ or } \partial_k v_k)$$

- Curl of a vector (denoted curl $\mathbf{v}$ or $\nabla \times \mathbf{v}$):

$$\nabla \times \mathbf{v} = \mathcal{E}_{ijk} \frac{\partial}{\partial x_i} v_j \mathbf{e}_k = \mathcal{E}_{ijk} v_{j,\,i} \mathbf{e}_k = \mathcal{E}_{ijk} (\partial_i v_j) \mathbf{e}_k$$

- The following relations hold:

$$\begin{aligned}
&\delta_{ii} = 3\\
&\delta_{ij}\delta_{jk} = \delta_{ik}\\
&\mathcal{E}_{ijk}\mathcal{E}_{ijm} = 2\delta_{km}\\
&\mathcal{E}_{ijk}\mathcal{E}_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\\
&\mathcal{E}_{ijk}\mathcal{E}_{ijk} = 6
\end{aligned}$$

- Typical identities:

  1. $(\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{2}\nabla(\mathbf{v} \cdot \mathbf{v}) - \mathbf{v} \times (\nabla \times \mathbf{v})$
  2. $\nabla \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{w} \cdot \nabla)\mathbf{v} + \mathbf{v} \cdot (\nabla \cdot \mathbf{w}) - \mathbf{w} \cdot (\nabla \cdot \mathbf{v}) - (\mathbf{v} \cdot \nabla)\mathbf{w}$

- Tensors:

$$\mathbf{e}_i \otimes \mathbf{e}_j = \text{tensor product of } \mathbf{e}_i \text{ and } \mathbf{e}_j$$

$$\mathbf{A} = \text{second-order tensor} = \left(\sum_{i,j}\right) A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = A_{ij}\ \mathbf{e}_i \otimes \mathbf{e}_j \qquad (A_{ij} = \mathbf{e}_i \cdot \mathbf{A}\mathbf{e}_j)$$

$$\mathbf{B} = \text{third-order tensor} = B_{ijk}\ e_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$$

Assignment 1

Vectors and Index Notation

1. Prove that:

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v}) - \mathbf{v}\times(\nabla\times\mathbf{v})$$

2. Prove that:

$$\nabla\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{w}\cdot\nabla)\mathbf{v} + \mathbf{v}(\nabla\cdot\mathbf{w}) - \mathbf{w}(\nabla\cdot\mathbf{v}) - (\mathbf{v}\cdot\nabla)\mathbf{w}$$

Vectors and Inner Product Spaces

3. Give a complete definition and a non-trivial example of

   (a) a real vector space

   (b) an inner product space

   (c) a linear transformation from a vector space U into a vector space V

Tensors

4. Let V be an inner product space. A tensor is a linear transformation from V into V. If $\mathbf{T}$ is a tensor, $\mathbf{Tv}$ denotes the image of the vector $\mathbf{v}$ in V:

$$\mathbf{Tv} = \mathbf{T}(\mathbf{v}) \in V$$

   Show that the class L(V,V) of all linear transformations of V into itself is also a vector space with vector addition and scalar multiplication defined as follows:

$$\begin{aligned} &\mathbf{S},\mathbf{T} \in L(V,V): \\ &\quad \mathbf{S}+\mathbf{T} = \mathbf{R} \Leftrightarrow \mathbf{Rv} = \mathbf{Sv} + \mathbf{Tv} \\ &\quad \alpha\mathbf{S} = \mathbf{R} \Leftrightarrow \mathbf{Rv} = \alpha(\mathbf{Sv}) \\ &\forall\,\mathbf{v} \in V,\ \forall\,\alpha \in \mathbb{R} \\ &(\mathbf{0} \in L(V,V):\ \mathbf{0v} = \mathbf{0} \in V) \end{aligned}$$

Tensor Product

5. The *tensor product* of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the tensor, denoted $\mathbf{a}\otimes\mathbf{b}$, that assigns to each vector $\mathbf{c}$ the vector $(\mathbf{b}\cdot\mathbf{c})\mathbf{a}$; that is

$$(\mathbf{a}\otimes\mathbf{b})\mathbf{c} = (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

   5.1 Show that $\mathbf{a}\otimes\mathbf{b}$ is a tensor and that

$$(\mathbf{a}\otimes\mathbf{b})^T = \mathbf{b}\otimes\mathbf{a}$$

$$(\mathbf{a}\otimes\mathbf{b})(\mathbf{c}\otimes\mathbf{d}) = (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\otimes\mathbf{d})$$

5.2 If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis ($\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, $1 \le i, j \le 3$), $\|\mathbf{e}_i\|^2 = \mathbf{e}_i \cdot \mathbf{e}_i = 1$), then show that

$$(\mathbf{e}_i \otimes \mathbf{e}_i)(\mathbf{e}_j \otimes \mathbf{e}_j) = \begin{cases} \mathbf{0}, & \text{if } i \neq j \\ \mathbf{e}_i \otimes \mathbf{e}_i, & \text{if } i = j \end{cases} = \text{“}\delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\text{”}$$

5.3 For an arbitrary tensor $\mathbf{A}$, and for the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$,

$$\mathbf{A} = \sum_{i,j} A_{ij}\ \mathbf{e}_i \otimes \mathbf{e}_j$$

where

$$A_{ij} = \mathbf{e}_i \cdot \mathbf{A}\mathbf{e}_j$$

The array $[A_{ij}]$ is the matrix characterizing $\mathbf{A}$ for this particular choice of a basis for $V(= \mathbb{R}^3)$. If $\mathbf{A}$ and $\mathbf{B}$ are two (“second-order”) tensors and $[A_{ij}]$, $[B_{ij}]$ are their matrices corresponding to a basis $\{e_1, e_2, e_3\}$ of V, define (construct) the rules of matrix algebra:

(a) $\mathbf{A} + \mathbf{B} = \mathbf{C} \Rightarrow [A_{ij}] + [B_{ij}] = [?]$

(b) $\mathbf{AB} = \mathbf{C}$ ($\mathbf{AB} = \mathbf{A} \circ \mathbf{B}$)

(c) $\mathbf{A0} = \mathbf{C}$ ($\mathbf{C} =?$) ($\mathbf{0} =$ the *zero element* of $L(V, V)$).

(d) $\mathbf{AC} = \mathbf{I}$ ($\mathbf{I}$ is the *identity tensor*: $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$, and $\mathbf{C} = \mathbf{A}^{-1}$)

(e) $\mathbf{A}^T = \mathbf{C}$ ($\mathbf{C} =?$) ($\mathbf{A}^T$ is the *transpose* of $\mathbf{A}$: it is the unique tensor such that $\mathbf{Av} \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{A}^T\mathbf{u}$, “$\cdot$” being the vector inner product in $\mathbb{R}^3$).

6. The inner product(“dot-product”) of two vectors $\mathbf{u}, \mathbf{v} \in V = \mathbb{R}^3$ is denoted $\mathbf{u}\cdot\mathbf{v}$. It is a symmetric, positive-definite, bilinear form on V. If $\mathbf{u} = \sum_{i=1}^3 \mathbf{u}_i\mathbf{e}_i$ and $\mathbf{v} = \sum_{i=1}^3 \mathbf{v}_i\mathbf{e}_i$, for an orthonormal basis $\{\mathbf{e}_i\}$, then $\mathbf{u}\cdot\mathbf{v} = \sum_{i=1}^3 u_i v_i$. Moreover, the (Euclidean) norm of $\mathbf{u}$ ($\mathbf{v}$) is $\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$ ($\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$).

The space $L(V, V)$ of second-order tensors can be naturally equipped with an inner product as well, and hence a norm. The construction is as follows:

i) The *trace* of the *tensor product* of two vectors $\mathbf{u}$ and $\mathbf{v}$ is the linear operation

$$\operatorname{tr}(\mathbf{u} \otimes \mathbf{v}) \overset{def}{=} \mathbf{u} \cdot \mathbf{v}$$

Likewise, the *trace* of a tensor $\mathbf{A} \in L(V, V)$ is defined by

$$\operatorname{tr}\mathbf{A} = \operatorname{tr}\left(\sum_{ij} A_{ij}\ \mathbf{e}_i \otimes \mathbf{e}_j\right) = \sum_{ij} A_{ij}\ \operatorname{tr}\mathbf{e}_i \otimes \mathbf{e}_j = \sum_{ij} A_{ij}\delta_{ij} = \sum_i A_{ii}$$

ii) The trace of the composition of two tensors $\mathbf{A}, \mathbf{B} \in L(V, V)$ is then

$$\mathrm{tr}(\mathbf{AB}) = \sum_{ij} A_{ij} B_{ji}$$

We denote

$$\mathbf{A} : \mathbf{B} = \mathrm{tr}(\mathbf{A}^T \mathbf{B}) = \sum_{ij} A_{ij} B_{ij}$$

(a) Show that $\mathbf{A} : \mathbf{B}$ (the operation ":") defines an inner product on $L(V, V)$

(b) Define the associated norm $\|\mathbf{A}\|$ of $\mathbf{A} \in L(V, V)$

(c) Show that

$$\mathrm{tr}\ \mathbf{A} = \mathrm{tr}\ \mathbf{A}^T$$

(d) Show that (trivially)

$$\mathbf{I} : \mathbf{A} = \mathrm{tr}\ \mathbf{A} \qquad (\mathbf{I} = \text{identity tensor})$$
$$(\mathbf{u} \otimes \mathbf{v}) : (\mathbf{q} \otimes \mathbf{p}) = (\mathbf{u} \cdot \mathbf{q})(\mathbf{v} \cdot \mathbf{p})$$

Assignment 2

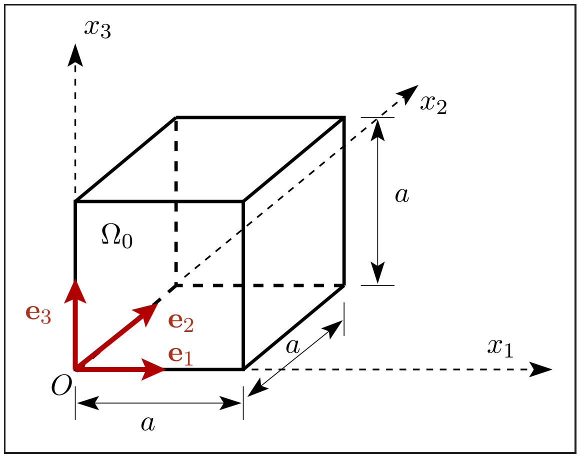

Figure 9.1: The cube $\Omega_0 = (0, a)^3$.

<u>**Kinematics of Continuous Media**</u>

1. The reference configuration of a deformable body $\mathcal{B}$ is the cube $\Omega_0 = (0, a)^3$ with the origin of the spatial and material coordinates at the corner, as shown in Fig 9.1. Consider a motion $\boldsymbol{\varphi}$ of the body defined by

$$\boldsymbol{\varphi}(\mathbf{X}) = \sum_{i=1}^{3} \varphi_i(\mathbf{X})\mathbf{e}_i$$

where

$$\varphi_1(\mathbf{X}) = X_1 + \frac{\Delta}{2}\left(\frac{X_2}{a}\right)^2$$
$$\varphi_2(\mathbf{X}) = X_2$$
$$\varphi_3(\mathbf{X}) = X_3$$

where Δ is a real number (a parameter possibly depending on time t).

For this motion,

(a) Sketch the deformed shape (i.e. sketch the current configuration) in the X_1–X_2 plane for $\Delta = a$.

Then compute the following:

(b) the displacement field $\mathbf{u}$

(c) the deformation gradient $\mathbf{F}$

(d) the deformation tensor $\mathbf{C}$

(e) the Green strain tensor $\mathbf{E}$

(f) the extensions $e_i, \ i = 1, 2, 3$

(g) $\sin\gamma_{12}$, where γ_{12} is the shear in the X_1–X_2 plane.

Determinants

2. Let $\mathbf{A} \in L(V, V)$ be a second order tensor with a matrix $[A_{ij}]$ relative to a basis $\{\mathbf{e}_i\}_{i=1}^3$ (i.e. the A_{ij} are components of $\mathbf{A}$ with respect to $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$). For $n = 3$, the determinant of $[A_{ij}]$ is defined by

$$\det[A_{ij}] = \frac{1}{6} \sum_{\substack{ijk \\ rst}} \mathcal{E}_{ijk}\mathcal{E}_{rst}A_{ir}A_{js}A_{kt}$$

where

$$\mathcal{E}_{ijk} \begin{cases} 1 & \text{if } \{i,j,k\} \text{ is an even permutation of } \{i,j,k\} \\ -1 & \text{if } \{i,j,k\} \text{ is an odd permutation of } \{i,j,k\} \\ 0 & \text{if } \{i,j,k\} \text{ is not a permutation of } \{i,j,k\} \end{cases}$$

(i.e. if at least two indices are equal).

The determinant of the tensor $\mathbf{A}$ is defined as the determinant of its matrix components A_{ij}:

$$\det \mathbf{A} = \det[A_{ij}]$$

This definition is *independent of the choice of basis* $\{e_i\}$ (i.e. $\det \mathbf{A}$ is a property of $\mathbf{A}$ *invariant* under changes of basis).

Show that

(a) $\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$ (for n=3 is sufficient)

(b) $\det \mathbf{A}^T = \det \mathbf{A}$

$$\boxed{\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}}$$

Take $n = 3$:

$$\det \mathbf{A} = \frac{1}{6}\,\mathcal{E}_{k\ell m}\mathcal{E}_{abc}A_{ka}A_{\ell b}A_{mc}$$
$$\det \mathbf{B} = \frac{1}{6}\,\mathcal{E}_{k\ell m}\mathcal{E}_{def}B_{kd}B_{\ell e}B_{mf}$$

But

$$\mathcal{E}_{k\ell m} \det \mathbf{A} = \mathcal{E}_{abc}A_{ka}A_{\ell b}A_{mc}$$
$$\mathcal{E}_{k\ell m} \det \mathbf{B} = \mathcal{E}_{def}B_{kd}B_{\ell e}A_{mc}$$

(for example: $\det \mathbf{A} = \mathcal{E}_{abc}A_{1a}A_{2b}A_{3c}$)

Note that $\mathcal{E}_{k\ell m}\mathcal{E}_{k\ell m} = 6$. Thus,

$$\begin{aligned}\mathcal{E}_{k\ell m} \det \mathbf{A}\, \mathcal{E}_{k\ell m} \det \mathbf{B} & \\ &= 6 \det \mathbf{A} \det \mathbf{B} \\ &= 6\, \mathcal{E}_{def}\mathcal{E}_{abc} A_{ka}B_{kd} \cdot A_{\ell b}B_{e\ell} \cdot A_{mc}B_{fm} \\ &= 6 \det \mathbf{AB}\end{aligned}$$

□

Cofactor and Inverse

3. For $\mathbf{A} \in L(V,V)$ and A_{ij} the components of $\mathbf{A}$ relative to a basis $\{\mathbf{e}_i\}$, $i = 1, 2, \ldots, n$, let A^1_{ij} be the elements of a matrix of order $(n-1)$ obtained by deleting the ith row and jth column of $[A_{ij}]$. The scalar

$$d_{ij} = (-1)^{i+j} \det[A^1_{ij}]$$

is called the (i,j)–*cofactor* of $[A_{ij}]$ and the matrix of cofactors = Cof $\mathbf{A} = [d_{ij}]$ is called the *cofactor matrix* of $\mathbf{A}$.

(a) Show that

$$\mathbf{A}(\text{Cof } \mathbf{A})^T = (\det \mathbf{A})\mathbf{I}$$

(it is sufficient to show this for $n = 3$ by construction)

(b) Show (trivially) that for $\mathbf{A}$ invertible,

$$\mathbf{A}^{-1} = (\det \mathbf{A})^{-1}(\text{Cof } \mathbf{A})^T$$

(c) For $n = 3$, show that

$$(\text{Cof } \mathbf{A})_{kt} = \frac{1}{2} \sum_{\substack{ijk \\ rst}} \mathcal{E}_{ijk}\mathcal{E}_{rst} A_{ir} A_{js}, \qquad 1 \le k,\, t \le 3$$

Orthogonal Transformation

4. A tensor $\mathbf{Q} \in L(V,V)$ is orthogonal if it preserves inner products, in the sense that

$$\mathbf{Qu} \cdot \mathbf{Qv} = \mathbf{u} \cdot \mathbf{v}$$

$\forall\, \mathbf{u}, \mathbf{v} \in V (\approx \mathbb{R}^3)$. Show that a necessary and sufficient condition that $\mathbf{Q}$ be orthogonal is that

$$\mathbf{QQ}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$$

or, equivalently,

$$\mathbf{Q}^T = \mathbf{Q}^{-1}$$

5. Confirm that $\det \mathbf{A}$ is an invariant of $\mathbf{A}$ in the following sense: if

$$\mathbf{A} = \sum_{i,j} A_{ij}\ \mathbf{e}_i \otimes \mathbf{e}_j = \sum_{i,j} \bar{A}_{ij}\ \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_j$$

where

$$\bar{\mathbf{e}}_i = \mathbf{Q}\mathbf{e}_i,\ \ i = 1, 2, 3,\ \ \mathbf{Q} = \text{an orthogonal tensor},$$

then

$$\det[A_{ij}] = \det[\bar{A}_{ij}]$$

Review of Vector and Tensor Calculus

6. Let U and V be finite-dimensional normed spaces and let $\mathbf{f}$ be a function from U into V. We say that $\mathbf{f}$ is differentiable at $\mathbf{u} \in U$ if there exists a linear functional $D\,\mathbf{f}$ on V such that

$$D\mathbf{f}(\mathbf{u}) \cdot \mathbf{v} = \lim_{\theta \to 0} \frac{1}{\theta}(\mathbf{f}(\mathbf{u} + \theta\mathbf{v}) - f(\mathbf{u}))$$

$D\mathbf{f}(\mathbf{u})$ is the derivative of f at $\mathbf{u}$.

Let φ be a scalar-valued function defined on the set $\mathcal{S}$ invertible tensors $\mathbf{A} \in L(U, U)$ (i.e. $\varphi : \mathcal{S} \to \mathbb{R}$) defined by

$$\varphi(\mathbf{A}) = \det \mathbf{A}$$

Show that (it is sufficient to consider the 3D case)

$$D\varphi(\mathbf{A}) : \mathbf{V} = (\det \mathbf{A})\mathbf{V}^T : \mathbf{A}^{-1}$$

Hint: Note that

$$\begin{aligned} \det(\mathbf{A} + \theta\mathbf{V}) &= \det((\mathbf{I} + \theta\mathbf{V}\mathbf{A}^{-1})\mathbf{A}) \\ &= \det \mathbf{A} \det(\mathbf{I} + \theta\mathbf{V}\mathbf{A}^{-1}) \end{aligned}$$

and that

$$\det(\mathbf{I} + \theta\mathbf{B}) = 1 + \theta \text{ tr } \mathbf{B} + 0(\theta^2)$$

7. Let Ω be an open set in $\mathbb{R}^3$ and φ be a smooth function mapping Ω into $\mathbb{R}$. The vector $\mathbf{g}$ with the property

$$D\varphi(\boldsymbol{x}) \cdot \mathbf{v} = \mathbf{g}(\boldsymbol{x}) \cdot \mathbf{v} \qquad \forall \mathbf{v} \in \mathbb{R}^3$$

is the gradient of $\mathbf{v}$ at point $\boldsymbol{x} \in \Omega$. We use the classical notation,

$$\mathbf{g}(\boldsymbol{x}) = \nabla\varphi(\boldsymbol{x})$$

Show that

$$\begin{aligned}\boldsymbol{\nabla}(\varphi\mathbf{v}) &= \varphi\boldsymbol{\nabla}\mathbf{v} + \mathbf{v}\otimes\nabla\varphi\\ \mathrm{div}(\varphi\mathbf{v}) &= \varphi\ \mathrm{div}\ \mathbf{v} + \mathbf{v}\cdot\nabla\varphi\\ \mathrm{div}(\mathbf{v}\otimes\mathbf{w}) &= \mathbf{v}\ \mathrm{div}\ \mathbf{w} + (\boldsymbol{\nabla}\mathbf{v})\mathbf{w}\end{aligned}$$

8. Let φ and $\mathbf{u}$ be C^2 scalar and vector fields. Show that
   (a) curl $\nabla\varphi = \mathbf{0}$
   (b) div curl $\mathbf{v} = 0$

9. Let Ω be an open, connected, smooth domain in $\mathbb{R}^3$ with boundary $\partial\Omega$. Let $\mathbf{n}$ be a unit exterior normal to $\partial\Omega$. Recall the Green (divergence) theorem,[1]

$$\int_\Omega \mathrm{div}\ \mathbf{v}\ dx = \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\ dA$$

for a vector field $\mathbf{v}$. Show that

$$\int_\Omega \mathrm{div}\ \mathbf{A}\ dx = \int_{\partial\Omega} \mathbf{A}\mathbf{n}\ dA$$

Hint: Note that for arbitrary vector $\mathbf{u}$,

$$\mathbf{u}\cdot\int_{\partial\Omega} \mathbf{A}\mathbf{n}\ dA = \int_{\partial\Omega} (\mathbf{A}^T\mathbf{u})\cdot\mathbf{n}\ dA$$

10. If (x, y, z) is a Cartesian coordinate system with origin at the corner of a cube $\Omega_0 = (0,1)^3$ and axes along edges of the cube, compute

$$\int_{\partial\Omega_0} \mathbf{v}\cdot\mathbf{n}\ dA_0$$

where $\partial\Omega_0$ is the exterior surface of Ω_0, $\mathbf{n}$ is the unit exterior normal vector, and $\mathbf{v}$ is the field, $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 + 3z\mathbf{e}_3$.

[1] div $\mathbf{v}$ = divergence of $\mathbf{v}$ = tr $\nabla\mathbf{v} = \sum_i \partial v_i/\partial x_i$ if $\mathbf{v} = \sum_i v_i\mathbf{e}_i$ div $\mathbf{A}$ is the unique vector field such that div $\mathbf{A}\cdot\mathbf{a} = \mathrm{div}(\mathbf{A}^T\mathbf{a})$ for all vector $\mathbf{a}$.

Assignment 3

1. The cube $(0,a)^3$ is the reference configuration Ω_0 of a body subjected to "simple shear"

$$u_1 = \frac{\Delta}{a} X_2, \quad u_2 = u_3 = 0\ , \qquad \Delta = \text{constant}$$

   The unit exterior normal to $\partial\Omega_0$ is $\mathbf{n}_0$. The vector $\mathbf{n}_0$ at boundary point $(a, a/2, a/2)$ is mapped into the unit exterior normal $\mathbf{n}$ on $\partial\Omega_t$. Compute $\mathbf{n}$ and sketch the deformed body.

2. Suppose Δ in Problem 1 above is a function of time t. Compute
   (a) the Lagrangian description of the velocity and acceleration fields,
   (b) the Eulerian descriptions of these fields,
   (c) $\mathbf{L} = \text{grad}\ \mathbf{v}$.

3. Show that $\mathbf{W}\mathbf{v} = \frac{1}{2}\boldsymbol{\omega} \times \mathbf{v}$

4. Suppose that at a material point $X \in \Omega_0$, the Green Strain tensor is given by

$$\mathbf{E} = \sum_{1 \le i,j \le 3} [E_{ij}]\, \mathbf{e}_i \otimes \mathbf{e}_j$$

   where $\{\mathbf{e}_i\}$ is an orthonormal basis in $\mathbb{R}^3$ and

$$[E_{ij}] = \begin{bmatrix} 3 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Determine:
   (a) the principal directions of $\mathbf{E}$,
   (b) the principal values of $\mathbf{E}$,
   (c) the transformation $\mathbf{Q}$ that maps $\{\mathbf{e}_i\}$ into the vectors defining the principal directions of $\mathbf{E}$,
   (d) the principal invariants of $\mathbf{C} = 2\mathbf{E} + \mathbf{I}$.

5. Recall that an invariant of $\mathbf{C}$ is any real-valued function $\mu(\mathbf{C})$ such that $\mu(\mathbf{C}) = \mu(\mathbf{A}^{-1}\mathbf{C}\mathbf{A})$ for all invertible matrices $\mathbf{A}$. Show that tr $\mathbf{C}$, tr Cof $\mathbf{C}$, and det $\mathbf{C}$ are invariants of $\mathbf{C}$.

6. Construct a detailed proof of the relations
   (a) $\mathbf{W}\mathbf{v} = \dfrac{1}{2}\boldsymbol{\omega} \times \mathbf{v}$, where $\boldsymbol{\omega} = \text{curl}\ \mathbf{v}$
   (b) $\dot{\det}\,\mathbf{F} = \det \mathbf{F}\ \text{div}\ \mathbf{v}$

Assignment 4

1. Reproduce the proof of Cauchy's Theorem for the existence of the stress tensor for the *two-dimensional* case ("plane stress") (to simplify geometric issues). Thus, for $\boldsymbol{\sigma}(\mathbf{n}) = \sigma_1(\mathbf{n})\mathbf{e}_1 + \sigma_2(\mathbf{n})\mathbf{e}_2$, show that $\exists\,\mathbf{T}$ such that $\boldsymbol{\sigma}(\mathbf{n}) = \mathbf{T}\mathbf{n}$.

2. The Cauchy stress tensor in a body $\mathcal{B}$ is

$$\mathbf{T}(\boldsymbol{x}, t) = T_{ij}(\boldsymbol{x}, t)\,\mathbf{e}_i \otimes \mathbf{e}_j$$

where

$$T_{ij}(\boldsymbol{x}, t) = e^{10-10t}\begin{bmatrix} 10000x_1^2 - 7000x_1x_2 & 7000x_1x_3 & 2000x_3^2 \\ 7000x_1x_3 & 3000x_2^2 & 100x_1 \\ 2000x_3^2 & 100x_1 & 1000x_3^2 \end{bmatrix}$$

   (a) At point $\boldsymbol{x} = (1,1,1)$ at time $t = 1$, compute the stress vector $\boldsymbol{\sigma}(\mathbf{n})$ in the direction $\mathbf{n} = n_i\mathbf{e}_i$, $n_i = 1/\sqrt{3}$, $i = 1, 2, 3$.

   (b) At $t = 1$, what is the total contact force on the plane surface $x_1 = 1$, $0 \le x_2 \le 1$, $0 \le x_3 \le 1$?

3. Let $\mathbf{T} = \mathbf{T}(\boldsymbol{x}, t)$ be the Cauchy stress at $\boldsymbol{x} \in \Omega_t$ at time t. If $\hat{\mathbf{n}}$ is a direction (a unit vector) such that

$$\mathbf{T}\hat{\mathbf{n}} = \sigma\hat{\mathbf{n}} \qquad (\hat{\mathbf{n}}^T\hat{\mathbf{n}} = 1)$$

then (in analogy with principal values and directions of $\mathbf{C}$), σ is a *principal stress* and $\hat{\mathbf{n}}$ is a *principal direction* of $\mathbf{T}$ (eigenvalues and eigenvectors of $\mathbf{T}$).

Continuing, let Γ be a plane through a point $\boldsymbol{x}$ with unit normal $\mathbf{n}$. The *normal stress* $\boldsymbol{\sigma}_n$ at $\boldsymbol{x}$ is

$$\boldsymbol{\sigma}_n = (\mathbf{n} \cdot \mathbf{T}\mathbf{n})\mathbf{n}$$

and the *shear stress* is

$$\boldsymbol{\sigma}_t = \mathbf{T}\mathbf{n} - \boldsymbol{\sigma}_n = \mathbf{T}\mathbf{n} - (\mathbf{n} \cdot \mathbf{T}\mathbf{n})\mathbf{n}$$

Show that if $\mathbf{n}$ were a principal direction of $\mathbf{T}$, then $\boldsymbol{\sigma}_t = \mathbf{0}$.

4. Newton's Law of action and reaction asserts that for each $\boldsymbol{x} \in \bar{\Omega}_t$ and each t,

$$\boldsymbol{\sigma}(\mathbf{n}, \boldsymbol{x}, t) = -\boldsymbol{\sigma}(-\mathbf{n}, \boldsymbol{x}, t)$$

Prove this law under the same assumptions as Cauchy's theorem.

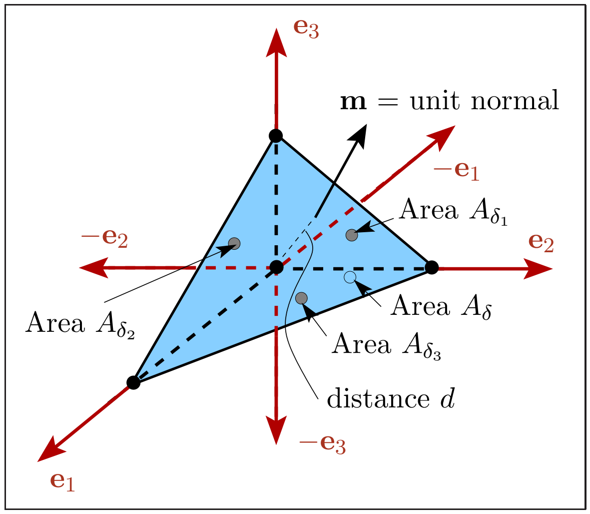

Figure 9.2: Tetrahedral element.

Hint:

(a) Construct the tetrahedral element as shown in Fig. 9.2.

(b) Compute total forces on tetrahedral element:

$$\frac{1}{A_\delta}\int_{A_d} \boldsymbol{\sigma}(\mathbf{m})\,dA + \frac{1}{A_{d_i}}\int_{A_{d_i}} \boldsymbol{\sigma}(-\mathbf{e}_i)dA + \text{body force } \mathcal{O}(d^3)$$

(c) Take limit as $d \to 0$; giving $\boldsymbol{\sigma}(\mathbf{n}, \boldsymbol{x}, t) = -\sum_{i=1}^{3}(\mathbf{m}\cdot\mathbf{e}_i)\boldsymbol{\sigma}(-\mathbf{e}_i, \boldsymbol{x}, t)$

(d) Take $\mathbf{m} = \mathbf{e}_1$, then $\mathbf{m} = \mathbf{e}_2$, then $\mathbf{m} = \mathbf{e}_3$ to establish that $\boldsymbol{\sigma}(\mathbf{e}_i, \boldsymbol{x}) = -\boldsymbol{\sigma}(-\mathbf{e}_i, \boldsymbol{x})$ (for any t). Then the assertion follows.

(e) Complete the proof of Cauchy's Theorem: Show $\mathbf{T}(\boldsymbol{x}) = \mathbf{T}(\boldsymbol{x})^T$. (This proof makes use of 1) the principle of balance of angular momentum, and 2) the equations of motion (div $\mathbf{T} + \mathbf{f} = \varrho\, d\mathbf{v}/dt$)).

Assignment 5

1. Let Γ be a material surface associated with the reference configuration: $\Gamma \subset \partial\Omega_t$. Let $\mathbf{g}$ be an applied force per unit area acting on $\Gamma(\mathbf{g} = \mathbf{g}(\boldsymbol{x}, t), \boldsymbol{x} \in \Gamma)$. The "traction" boundary condition on Γ at each $\boldsymbol{x} \in \Gamma$ is

$$\mathbf{Tn} = \mathbf{g}$$

Show that

$$\mathbf{FSn}_0 = \mathbf{g}_0 \qquad \text{on } \boldsymbol{\varphi}^{-1}(\Gamma)$$

where $\mathbf{n}_0$ is the unit exterior normal to Γ_0 ($\Gamma = \boldsymbol{\varphi}(\Gamma_0)$) and

$$\mathbf{g}_0(\mathbf{X}, t) = \det \mathbf{F}(\mathbf{X}) \|\mathbf{F}^{-T}(\mathbf{X})\mathbf{n}_0\| \; \mathbf{g}(\boldsymbol{x})$$

2. Consider an Eulerian description of the flow of a fluid in a region of $\mathbb{R}^3$. The flow is characterized by the triple $(\mathbf{v}, \varrho, \mathbf{T})$ = (velocity field, density field, Cauchy stress field). The flow is said to be *potential* if the velocity is derivable as the gradient of a scalar field φ:

$$\mathbf{v} = \text{grad } \varphi$$

The body force field acting on the fluid is said to be *conservative* if there is also a potential U such that

$$\mathbf{f} = -\varrho \text{ grad } U$$

The special case in which the stress $\mathbf{T}$ is of the form

$$\mathbf{T} = -p\,\mathbf{I}$$

where p is a scalar field and $\mathbf{I}$ is the unit tensor, is called a *pressure* field (p = the fluid pressure or "hydrostatic" pressure).

   (a) Show that for potential flow, a pressure field $\mathbf{T} = p\,\mathbf{I}$, and conservative body forces, the momentum equations imply that

$$\text{grad}\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + U\right) + \frac{1}{\varrho}\text{ grad } p = \mathbf{0}$$

   This is Bernoulli's Equation for potential flow.

   Hint: show that

   1) div $\mathbf{T} = -$grad p,
   2) $d\mathbf{v}/dt = \partial\mathbf{v}/\partial t + \frac{1}{2}$ grad $\mathbf{v}\cdot\mathbf{v}$

(b) If the motion is steady (i.e. if $\partial \mathbf{v}/\partial t = 0$, ($\mathbf{v}(\boldsymbol{x}, t)$ is invariant with respect to t)), and $\partial \mathbf{f}/\partial t = 0$, so $\partial U/\partial t = 0$, then the equations of part (a) reduce to

$$\mathbf{v} \cdot \text{grad}\left(\frac{1}{2}\mathbf{v} \cdot \mathbf{v} + U\right) + \frac{1}{\varrho}\mathbf{v} \cdot \text{grad}\ \mathbf{v} = \mathbf{0}$$

(Notice that $\mathbf{v} \cdot d\mathbf{v}/dt = \mathbf{v} \cdot \text{grad}(\frac{1}{2}\mathbf{v} \cdot \mathbf{v})$).

3. Let $\mathbf{u} = \mathbf{u}(\mathbf{X}, t) = \boldsymbol{\varphi}(\mathbf{X}, t) - \mathbf{X}$ be the displacement field. Define the quantity

$$\psi = \frac{1}{2}\int_{\Omega_0} \varrho_0 \mathbf{u} \cdot \mathbf{u}\, dX$$

Show that (if $\mathbf{f}_0 = \mathbf{0}$),

$$\ddot{\psi} = \int_{\Omega_0} \varrho_0 \dot{\mathbf{u}} \cdot \dot{\mathbf{u}}\, dX - \int_{\Omega_0} \mathbf{P} : \boldsymbol{\nabla}\mathbf{u}\, dX + \int_{\partial\Omega_0} \mathbf{u} \cdot \mathbf{P}\mathbf{n}_0\, dA_0$$

4. A cylindrical rubber plug 1 cm in diameter and 1 cm long (see Fig. 9.3) is glued to a rigid foundation. Then it is pulled by external forces so that the flat cylindrical upper face $\Gamma_0 = \{(X_1, X_2, X_3) :\ X_3 = 1,\ (X_1^2 + X_2^2)^{1/2} \leq 1/2\}$ is squeezed to a flat circular face Γ of diameter 1/4 cm with normal $\mathbf{n} = \mathbf{e}_2$, as shown at position $\boldsymbol{x} = \boldsymbol{x}^*$. Suppose that the stress vector at $\boldsymbol{x} = \boldsymbol{x}^*$ is uniform and normal to Γ:

$$\boldsymbol{\sigma}(\mathbf{n}, \boldsymbol{x}^*) = \boldsymbol{\sigma}(\mathbf{e}_2, \boldsymbol{x}^*) = 1000\, \mathbf{e}_2 \text{ kg/cm}^2 \qquad \boldsymbol{x}^* \in \Gamma$$

Suppose that the corresponding Piola-Kirchhoff stress $\mathbf{p}_0 = \mathbf{P}\,\mathbf{n}_0$ is uniform on Γ_0 and normal to Γ_0.

(a) Determine the Piola-Kirchhoff stress vector $\mathbf{p}_0 = \mathbf{P}\mathbf{n}_0$ on Γ_0.

(b) Determine one possible tensor Cof $\mathbf{F}(\mathbf{X}_1, X_2, 1)$ for this situation.

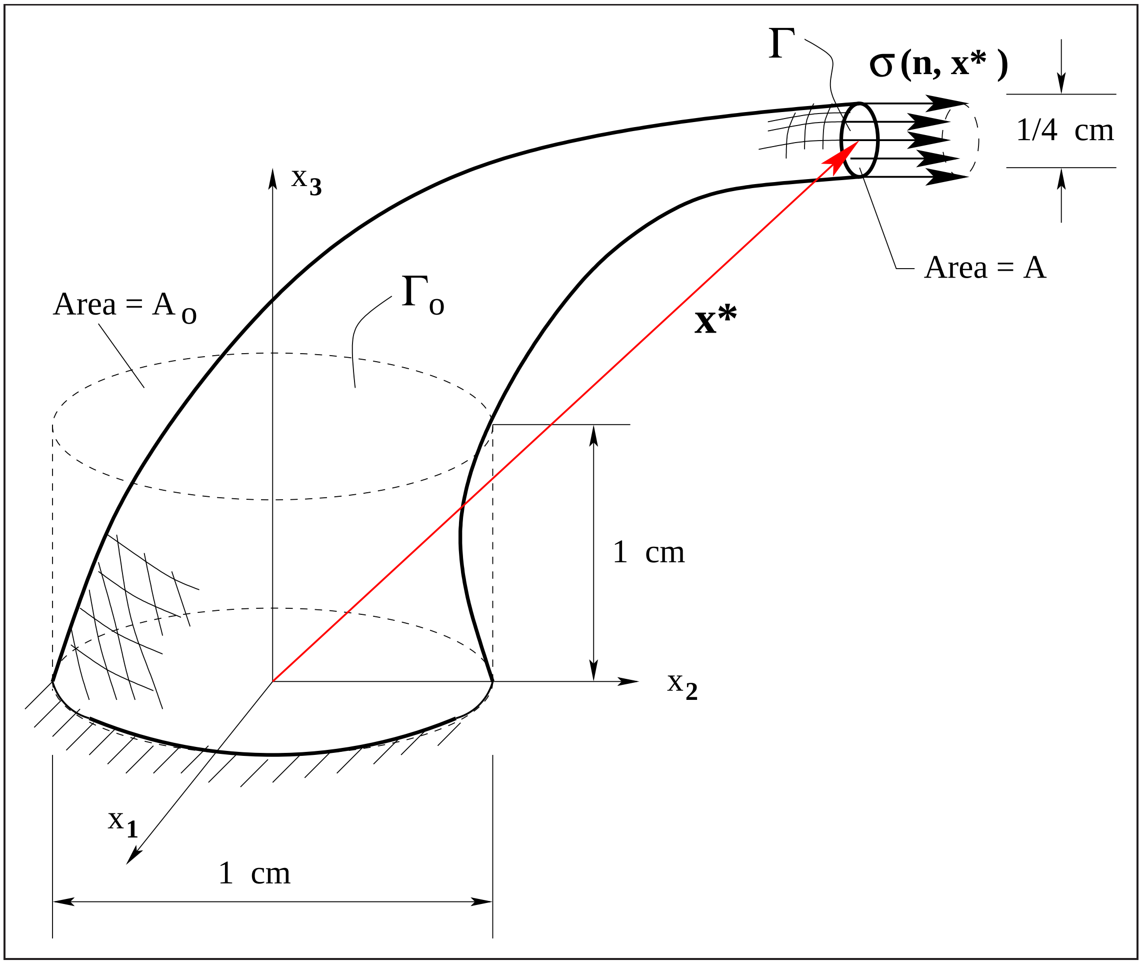

Figure 9.3: Illustrative sketch for question 4 of assignment 5.

Bibliography

[1] Ciarlet, P.G. **Mathematical Elasticity, Volume 1: Three-Dimensioned Elasticity**, North-Holland, Amsterdam, 1988.

[2] Eringen, A.C. **Nonlinear Theory of Continuous Media**, McGraw-Hill, New York, 1962.

[3] Gurtin, M.E. **An Introduction to Continuum Mechanics**, Academic Press, New York, 1981.

[4] Oden, J.T. **Finite Elements of Nonlinear Continua**, McGraw-Hill, New York,1972. Dover Edition, 2006.

[5] Truesdell, C. and Noll, W. **The Non-Linear Field Theories of Mechanics**, Handbuck der Physik, Vol. III/3, Springer, Berlin, 1965.

[6] Truesdell, C. and Toupin, R. **The Classical Field Theories**, Handbuck der Physik, Vol. III/1, Springer, Berlin, 1960.