Appendix A

Vector Operator in Matrix Form

A.1 Introduction

Equations of motion usually involve a time derivative ($\frac{\partial}{\partial t}$) and a spacial derivative ($\frac{\partial}{\partial x_i}$). It could be complicated to formulate and solve an equation of motion in three directions. Equations of motion will be easier to handle and solve if a spacial derivative is written in matrix form.

This appendix collects some useful vector and differential operators for mechanical aether theory. Operators are divided into three parts: Part A shows some useful vector operators in matrix form. Part B shows some useful differential operators in matrix form. Part C shows a special case of Part B when the differential operators are acting on wave form function. When differential operators in Part B are acting on wave form vector, differential operators change to constant numbers, and the differential equations of motion change to algebra equations.

This appendix provides some mathematical backgrounds and some useful formations of operators for solving equations of motion of aether particles.

In order to elevate the usage of the reference table, this appendix is organized in a different form from the other sections. All formulas are listed at the beginning of each part followed by remarks to provide examples and derivations of the formulas.

A.2 Part A: Vector Operators in Matrix Form

A vector ($\vec{A}$) is represented by 3 components (a_1, a_2, a_3) of 3 unit length base vectors $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ in Cartesian coordinate system, that is $\vec{A} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3$, or in matrix form as $\vec{A} = \langle \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3 \rangle \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \langle a_1 \quad a_2 \quad a_3 \rangle \begin{Bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{Bmatrix}$.

To simplify the formulation of equations of motion, the base vectors are neglected, and the vector $\vec{A}$ is represented by a column matrix $\{A\} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$, or a row matrix $\langle A \rangle = \langle a_1 \quad a_2 \quad a_3 \rangle$, or a square anti-symmetric matrix $[A] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$. The vector $\{A\}$ or $\langle A \rangle$ is the dual vector of the anti-symmetric matrix $[A]^T = -[A]$. (Another different relationship between the dual vector and the dual anti-symmetric matrix is $[A]$ defined as the dual matrix of $\vec{a}$, a difference in sign).

Therefore we have matrix forms for

A1: Vector: $\vec{A} \to \{A\}$ or $\langle A \rangle$ or $[A]$

A2: Dot product: $\vec{A} \cdot \vec{B} \to \langle A \rangle \{B\}$

A3: Cross product: $\vec{A} \times \vec{B} \to [A]\{B\}$ or $\langle A \rangle [B]$

A4: Dyac: $\vec{A} \otimes \vec{B} \to \{A\} \langle B \rangle$

Remark on A2 : Dot product with vector components,

$$
\begin{aligned}
\vec{A} \cdot \vec{B} &= \langle a_1 \quad a_2 \quad a_3 \rangle \begin{Bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{Bmatrix} \cdot \langle \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3 \rangle \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \\
&= \langle a_1 \quad a_2 \quad a_3 \rangle \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 & \vec{e}_1 \cdot \vec{e}_3 \\ \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 & \vec{e}_2 \cdot \vec{e}_3 \\ \vec{e}_3 \cdot \vec{e}_1 & \vec{e}_3 \cdot \vec{e}_2 & \vec{e}_3 \cdot \vec{e}_3 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \\
&= \langle a_1 \quad a_2 \quad a_3 \rangle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \langle a_1 \quad a_2 \quad a_3 \rangle \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \\
&= \langle A \rangle \{B\}
\end{aligned}
\tag{A.1}
$$

Remark on A3 : Cross product with vector components,

$$
\begin{aligned}
\vec{A}\times\vec{B} &= \langle a_1 \quad a_2 \quad a_3\rangle \begin{Bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{Bmatrix} \times \langle \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3\rangle \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \\
&= \langle a_1 \quad a_2 \quad a_3\rangle \begin{bmatrix} \vec{e}_1\times\vec{e}_1 & \vec{e}_1\times\vec{e}_2 & \vec{e}_1\times\vec{e}_3 \\ \vec{e}_2\times\vec{e}_1 & \vec{e}_2\times\vec{e}_2 & \vec{e}_2\times\vec{e}_3 \\ \vec{e}_3\times\vec{e}_1 & \vec{e}_3\times\vec{e}_2 & \vec{e}_3\times\vec{e}_3 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \\
&= \langle a_1 \quad a_2 \quad a_3\rangle \begin{bmatrix} \vec{0} & \vec{e}_3 & -\vec{e}_2 \\ -\vec{e}_3 & \vec{0} & \vec{e}_1 \\ \vec{e}_2 & -\vec{e}_1 & \vec{0} \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \\
&= \langle \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3\rangle \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \rightarrow [A]\{B\} \qquad \text{(A.2)} \\
&= \langle a_1 \quad a_2 \quad a_3\rangle \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{Bmatrix} \rightarrow \langle A\rangle [B] \qquad \text{(A.3)}
\end{aligned}
$$

Note that we do not use the following form to represent the cross product since it is not consistent with the other operators.

$$
\vec{A}\times\vec{B} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \text{(A.4)}
$$

Remark on A4 : Dyac with vector components,

$$
\begin{aligned}
\vec{A}\otimes\vec{B} &= \langle \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3\rangle \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \langle b_1 \quad b_2 \quad b_3\rangle \begin{Bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{Bmatrix} \\
&= \langle \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3\rangle \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix} \begin{Bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{Bmatrix} \rightarrow \{A\}\langle B\rangle \qquad \text{(A.5)}
\end{aligned}
$$

Note that $\vec{A}\cdot\vec{B}$ is a scalar. $\vec{A}\times\vec{B}$ is a vector. $\vec{A}\otimes\vec{B}$ is represented by a square matrix which has two base vectors associated with its rows and columns.

A.3 Part B: Differential Operators in Matrix Form

Similar to part A, we have matrix forms for differential operators ∇ as follows.

$$\{\nabla\} = \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \tag{A.6}$$

$$\langle\nabla\rangle = \langle \frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2} \quad \frac{\partial}{\partial x_3} \rangle \tag{A.7}$$

$$[\nabla] = \begin{bmatrix} 0 & \frac{-\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{-\partial}{\partial x_1} \\ \frac{-\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix} \tag{A.8}$$

Therefore we have matrix forms for

B1: Differential vector: $\vec{\nabla} \to \{\nabla\}$ or $\langle\nabla\rangle$ or $[\nabla]$

B2: Divergence: $\vec{\nabla} \cdot \vec{u} \to \langle\nabla\rangle \{u\}$

B3: Curl: $\vec{\nabla} \times \vec{u} \to [\nabla] \{u\}$

B4: Dyac: $\vec{\nabla} \otimes \vec{u} \to \{\nabla\} \langle u\rangle$

B5: Gradient: $\vec{\nabla}\phi \to \{\nabla\} \phi$

B6: Useful relation 1: $\langle\nabla\rangle \{\nabla\} = \nabla^2$

B7: Useful relation 2: $[\nabla] \{\nabla\} = \{0\}$ and $\langle\nabla\rangle [\nabla] = \langle 0\rangle$

B8: Useful relation 3: $[\nabla] [\nabla] = \{\nabla\} \langle\nabla\rangle - \langle\nabla\rangle \{\nabla\} [I] = \{\nabla\} \langle\nabla\rangle - \nabla^2 [I]$

B9: Useful relation 4: $[\nabla] [\nabla] [\nabla] = -\langle\nabla\rangle \{\nabla\} [\nabla] = -\nabla^2 [\nabla]$

B10: Laplace: $\nabla^2\vec{u} \to \langle\nabla\rangle \{\nabla\} \{u\}$ and $\nabla^2\phi \to \langle\nabla\rangle \{\nabla\} \phi$

B11: Curl divergence: $\vec{\nabla} \times \vec{\nabla} \cdot \vec{u} = \vec{\nabla} \cdot \vec{\nabla} \times \vec{u} = 0 \to \langle\nabla\rangle [\nabla] \{u\} = 0$

B12: Double curl: $\vec{\nabla} \times \vec{\nabla} \times \vec{u} = \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) - \nabla^2\vec{u} \to$
$[\nabla] [\nabla] \{u\} = \{\nabla\} \langle\nabla\rangle \{u\} - \nabla^2 \{u\}$

B13: Triple curl: $\vec{\nabla} \times \vec{\nabla} \times \vec{\nabla} \times \vec{u} = -\nabla^2\vec{\nabla} \times \vec{u} \to$
$[\nabla] [\nabla] [\nabla] \{u\} = -\nabla^2 [\nabla] \{u\}$

Remark on B2:

$$\vec{\nabla} \cdot \vec{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \langle \frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2} \quad \frac{\partial}{\partial x_3} \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \langle\nabla\rangle \{u\} \tag{A.9}$$

Remark on B3:

$$\vec{\nabla} \times \vec{u} = \langle \tfrac{\partial}{\partial x_1} \quad \tfrac{\partial}{\partial x_2} \quad \tfrac{\partial}{\partial x_3} \rangle \begin{Bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{Bmatrix} \times \langle \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3 \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \langle \tfrac{\partial}{\partial x_1} \quad \tfrac{\partial}{\partial x_2} \quad \tfrac{\partial}{\partial x_3} \rangle \begin{bmatrix} \vec{e}_1 \times \vec{e}_1 & \vec{e}_1 \times \vec{e}_2 & \vec{e}_1 \times \vec{e}_3 \\ \vec{e}_2 \times \vec{e}_1 & \vec{e}_2 \times \vec{e}_2 & \vec{e}_2 \times \vec{e}_3 \\ \vec{e}_3 \times \vec{e}_1 & \vec{e}_3 \times \vec{e}_2 & \vec{e}_3 \times \vec{e}_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \langle \tfrac{\partial}{\partial x_1} \quad \tfrac{\partial}{\partial x_2} \quad \tfrac{\partial}{\partial x_3} \rangle \begin{bmatrix} \vec{0} & \vec{e}_3 & -\vec{e}_2 \\ -\vec{e}_3 & \vec{0} & \vec{e}_1 \\ \vec{e}_2 & -\vec{e}_1 & \vec{0} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \langle \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3 \rangle \begin{bmatrix} 0 & \frac{-\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{-\partial}{\partial x_1} \\ \frac{-\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \rightarrow [\nabla] \{u\} \tag{A.10}$$

$$= \langle \tfrac{\partial}{\partial x_1} \quad \tfrac{\partial}{\partial x_2} \quad \tfrac{\partial}{\partial x_3} \rangle \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{Bmatrix} \rightarrow \langle \nabla \rangle [u] \tag{A.11}$$

Remark on B4:

$$\vec{\nabla} \otimes \vec{u} = \langle \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3 \rangle \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \langle u_1 \quad u_2 \quad u_3 \rangle \begin{Bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{Bmatrix}$$

$$= \langle \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3 \rangle \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{Bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{Bmatrix} \rightarrow \{\nabla\} \langle u \rangle \tag{A.12}$$

Remark on B5:

$$\vec{\nabla} \phi = \langle \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3 \rangle \begin{Bmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} \end{Bmatrix} = \langle \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3 \rangle \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \phi \rightarrow \{\nabla\} \phi \tag{A.13}$$

Remark on B6:

$$\langle \nabla \rangle \{\nabla\} = \langle \tfrac{\partial}{\partial x_1} \quad \tfrac{\partial}{\partial x_2} \quad \tfrac{\partial}{\partial x_3} \rangle \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) = \nabla^2 \tag{A.14}$$

Remark on B7:

$$[\nabla]\{\nabla\} = \begin{bmatrix} 0 & \frac{-\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{-\partial}{\partial x_1} \\ \frac{-\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{A.15}$$

$$\langle\nabla\rangle[\nabla] = \langle \frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2} \quad \frac{\partial}{\partial x_3} \rangle \begin{bmatrix} 0 & \frac{-\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{-\partial}{\partial x_1} \\ \frac{-\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix} = \langle 0 \quad 0 \quad 0 \rangle \tag{A.16}$$

Remark on B8:

$$\begin{aligned} [\nabla][\nabla] &= \begin{bmatrix} 0 & \frac{-\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{-\partial}{\partial x_1} \\ \frac{-\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{-\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{-\partial}{\partial x_1} \\ \frac{-\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} & \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} & -\frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_1^2} & \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_2} & -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_3} \end{bmatrix} - (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \langle \frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2} \quad \frac{\partial}{\partial x_3} \rangle - \langle \frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2} \quad \frac{\partial}{\partial x_3} \rangle \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \{\nabla\}\langle\nabla\rangle - \langle\nabla\rangle\{\nabla\}[I] = \{\nabla\}\langle\nabla\rangle - \nabla^2[I] \end{aligned} \tag{A.17}$$

Remark on B9:

Use B8 to replace the two curl operators and use B7 to get

$$\begin{aligned} [\nabla][\nabla][\nabla] &= (\{\nabla\}\langle\nabla\rangle - \langle\nabla\rangle\{\nabla\}[I])[\nabla] \\ &= \{\nabla\}\langle\nabla\rangle[\nabla] - \langle\nabla\rangle\{\nabla\}[\nabla] \\ &= -\langle\nabla\rangle\{\nabla\}[\nabla] = -\nabla^2[\nabla] \end{aligned} \tag{A.18}$$

Remark on B10-B13:

The results can be obtained by using the relations in B6-B9.

A.4 Part C: Differential Operators for Harmonic Function

Differential operators in Part B can be reduced to algebra operators when they act on harmonic function as

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \exp(j(\langle k_1 \quad k_2 \quad k_3 \rangle \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} - \omega t)) \tag{A.19}$$

C1: Time derivative operator: $\frac{\partial}{\partial t}\vec{u} = -j\omega\vec{u} \quad \rightarrow \quad \frac{\partial}{\partial t}\{u\} = -j\omega\{u\}$

C2: Divergence: $\vec{\nabla} \cdot \vec{u} = j\vec{k} \cdot \vec{u} \quad \rightarrow \quad \langle \nabla \rangle \{u\} = j\langle k \rangle \{u\}$

C3: Curl: $\vec{\nabla} \times \vec{u} = j\vec{k} \times \vec{u} \quad \rightarrow \quad [\nabla]\{u\} = j[k]\{u\}$

C4: Dyac: $\vec{\nabla} \otimes \vec{u} = j\vec{k} \otimes \vec{u} \quad \rightarrow \quad \{\nabla\}\langle u \rangle = j\{k\}\langle u \rangle$

C5: Gradient: $\vec{\nabla}\phi = j\vec{k}\phi \quad \rightarrow \quad \{\nabla\}\phi = j\{k\}\phi$

C6: Useful relation 1: $\langle k \rangle \{k\} = k^2$

C7: Useful relation 2: $[k]\{k\} = \{0\}$ and $\langle k \rangle [k] = \langle 0 \rangle$

C8: Useful relation 3: $[k][k] = \{k\}\langle k \rangle - \langle k \rangle \{k\}[I] = \{k\}\langle k \rangle - k^2[I]$

C9: Useful relation 4: $[k][k][k] = -\langle k \rangle \{k\}[k] = -k^2[k]$

C10: Laplace: $\nabla^2\vec{u} = -k^2\vec{u} \quad \rightarrow \quad \nabla^2\{u\} = -k^2\{u\}$

C11: Double curl: $\vec{\nabla} \times \vec{\nabla} \times \vec{u} = -\vec{k} \times \vec{k} \times \vec{u} \quad \rightarrow$
$[\nabla][\nabla]\{u\} = -[k][k]\{u\} = -\{k\}\langle k \rangle \{u\} + k^2\{u\}$

C12: Triple curl: $\vec{\nabla} \times \vec{\nabla} \times \vec{\nabla} \times \vec{u} = -j\vec{k} \times \vec{k} \times \vec{k} \times \vec{u} \quad \rightarrow$
$[\nabla][\nabla][\nabla]\{u\} = -j[k][k][k]\{u\} = jk^2[k]\{u\}$

Remark on C2:

$$
\begin{aligned}
\langle\nabla\rangle\{u\} &= \langle \tfrac{\partial}{\partial x_1} \quad \tfrac{\partial}{\partial x_2} \quad \tfrac{\partial}{\partial x_3} \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \\
&= \langle \tfrac{\partial}{\partial x_1} \quad \tfrac{\partial}{\partial x_2} \quad \tfrac{\partial}{\partial x_3} \rangle \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \exp(j(\langle k_1 \quad k_2 \quad k_3 \rangle \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} - \omega t)) \\
&= \langle a_1 \quad a_2 \quad a_3 \rangle \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \exp(j(\langle k_1 \quad k_2 \quad k_3 \rangle \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} - \omega t)) \\
&= \langle a_1 \quad a_2 \quad a_3 \rangle \begin{Bmatrix} jk_1 \\ jk_2 \\ jk_3 \end{Bmatrix} \exp(j(\langle k_1 \quad k_2 \quad k_3 \rangle \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} - \omega t)) \\
&= j\langle k_1 \quad k_2 \quad k_3 \rangle \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \exp(j(\langle k_1 \quad k_2 \quad k_3 \rangle \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} - \omega t)) \\
&= j\langle k_1 \quad k_2 \quad k_3 \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = j\langle k\rangle\{u\} \qquad \text{(A.20)}
\end{aligned}
$$

Remark on C3:

$$
\begin{aligned}
[\nabla]\{u\} &= \begin{bmatrix} 0 & \frac{-\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{-\partial}{\partial x_1} \\ \frac{-\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \exp(j(\langle k_1 \quad k_2 \quad k_3 \rangle \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} - \omega t)) \\
&= j \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \exp(j(\langle k_1 \quad k_2 \quad k_3 \rangle \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} - \omega t)) \\
&= j \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = j\,[k]\{u\} \qquad \text{(A.21)}
\end{aligned}
$$

Remark on C4:

$$
\begin{aligned}
\{\nabla\}\langle u\rangle &= \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \langle u_1 \quad u_2 \quad u_3\rangle \\
&= \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \langle a_1 \quad a_2 \quad a_3\rangle \exp\left(j\left(\langle k_1 \quad k_2 \quad k_3\rangle \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} - \omega t\right)\right) \\
&= j \begin{Bmatrix} k_1 \\ k_2 \\ k_3 \end{Bmatrix} \langle a_1 \quad a_2 \quad a_3\rangle \exp\left(j\left(\langle k_1 \quad k_2 \quad k_3\rangle \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} - \omega t\right)\right) \\
&= j \begin{Bmatrix} k_1 \\ k_2 \\ k_3 \end{Bmatrix} \langle u_1 \quad u_2 \quad u_3\rangle = j\{k\}\langle u\rangle
\end{aligned} \tag{A.22}
$$

Remark on C5:

$$
\begin{aligned}
\{\nabla\}\phi &= \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \phi \\
&= \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} q \exp\left(j\left(\langle k_1 \quad k_2 \quad k_3\rangle \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} - \omega t\right)\right) \\
&= j \begin{Bmatrix} k_1 \\ k_2 \\ k_3 \end{Bmatrix} q \exp\left(j\left(\langle k_1 \quad k_2 \quad k_3\rangle \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} - \omega t\right)\right) \\
&= j \begin{Bmatrix} k_1 \\ k_2 \\ k_3 \end{Bmatrix} \phi = j\{k\}\phi
\end{aligned} \tag{A.23}
$$

Remark on C6:

$$
\langle k\rangle\{k\} = \langle k_1 \quad k_2 \quad k_3\rangle \begin{Bmatrix} k_1 \\ k_2 \\ k_3 \end{Bmatrix} = (k_1^2 + k_2^2 + k_3^2) = k^2 \tag{A.24}
$$

Remark on C7:

$$[k]\,\{k\} = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} \begin{Bmatrix} k_1 \\ k_2 \\ k_3 \end{Bmatrix} = \begin{Bmatrix} k_2k_3 - k_3k_2 \\ k_3k_1 - k_1k_3 \\ k_1k_2 - k_2k_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{A.25}$$

$$\langle k\rangle\,[k] = \langle k_1 \quad k_2 \quad k_3\rangle \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} = \langle 0 \quad 0 \quad 0\rangle \tag{A.26}$$

Remark on C8:

$$\begin{aligned}
[k]\,[k] &= \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -k_2^2 - x_3^2 & k_1k_2 & k_1k_3 \\ k_2k_1 & -k_3^2 - k_1^2 & k_2k_3 \\ k_3k_1 & k_3k_2 & -k_1^2 - k_2^2 \end{bmatrix} \\
&= \begin{bmatrix} k_1k_1 & k_1k_2 & k_1k_3 \\ k_2k_1 & k_2k_2 & k_2k_3 \\ k_3k_1 & k_3k_2 & k_3k_3 \end{bmatrix} - (k_1^2 + k_2^2 + k_3^2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{Bmatrix} k_1 \\ k_2 \\ k_3 \end{Bmatrix} \langle k_1 \quad k_2 \quad k_3\rangle - \langle k_1 \quad k_2 \quad k_3\rangle \begin{Bmatrix} k_1 \\ k_2 \\ k_3 \end{Bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \{k\}\,\langle k\rangle - \langle k\rangle\,\{k\}\,[I] = \{k\}\,\langle k\rangle - k^2\,[I]
\end{aligned} \tag{A.27}$$

Remark on C9:

Use C8 to replace the two cross operators and use C7 to get:

$$\begin{aligned}
[k]\,[k]\,[k] &= (\{k\}\,\langle k\rangle - \langle k\rangle\,\{k\}\,[I])\,[k] \\
&= \{k\}\,\langle k\rangle\,[k] - \langle k\rangle\,\{k\}\,[k] \\
&= -\langle k\rangle\,\{k\}\,[k] = -k^2\,[k]
\end{aligned} \tag{A.28}$$

Remark on C10-C12:

The results can be obtained by using the relations in C6-C9.

ISBN: 0986388408
ISBN-13: 978-0986388408
First Edition