The Principle of Duality in Clifford Algebra and Projective Geometry*

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Abstract

A completely dual approach to Clifford algebra is presented in this article. It leads to the introduction of two new products, the dual geometric product $*$ and the dual inner product $\circ$, and sheds new light on the duality relation between the progressive and regressive outer products of the Clifford algebra. On the firm base of this dual approach, the projective principle of duality is formulated in this completeness for the first time in the language of Clifford algebra. In order to provide mathematical concepts which are close to possible applications in theoretical physics, section 5 is devoted to projective coordinate systems. The incidence relations between the primitive geometric forms, the linear complex and Desargues' theorem are discussed as well.

1 Introduction

The present article originates in the quest of the role played by the projective principle of duality in physics. Projective geometry and its principle of duality was applied to mechanics in the 19th century by several mathematicians and physicists ([Zi]). One of the first among them was the French geometer M. Chasles who considered the principle of duality as an important instrument not only in geometry but also in mechanics: "L'application des mêmes idées de dualité peut s'étendre à la Mécanique. En effet, l'élément primitif des corps auquel on applique d'abord les premiers principes de cette science, est, comme dans la Géométrie ancienne, le *point* mathématique. Ne sommes-nous pas autorisés à penser, maintenant, qu'en prenant le *plan* pour l'élément de l'étendue, et non plus le *point*, on sera conduit à d'autres doctrines, faisant pour ainsi dire une

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nouvelle science? Et s'il existe un principe unique pour passer de cette science à l'ancienne, comme le théorème de Géométrie qui établit la corrélation des propriétés de l'étendue figurée, ce principe sera la base d'une *dualité* semblable, dans la science du mouvement des corps." ([Zi], p. 21) One of the most advanced approaches to 'geometric mechanics'—a notion coined by R. Ziegler in [Zi] for the application of methods from projective geometry to mechanics—certainly is the dissertation of F. Lindemann [Li].

Nevertheless, the significance of the principle of duality in modern physics, especially in relativistic quantum mechanics, has not yet been worked out. Since Clifford algebra is an excellent language for projective geometry and physics as well, it was necessary to find the equivalent formulation of the principle of duality in Clifford algebra. In this article we limit our intention to the formulation of the projective principle of duality in Clifford algebra. It will serve as the mathematical backbone in further investigations.

The Clifford algebra was introduced by the English mathematician W. K. Clifford (1878) in his analysis of Grassmann's papers. W. K. Clifford called the new algebra 'geometric algebra'. In addition, Clifford algebra appears to be an ideal tool for the formulation of geometric concepts. For this reason we will use the term 'geometric algebra' instead of 'Clifford algebra' from now on throughout this article.

After a short introduction to the concepts of geometric algebra in section 2, we describe the completely dual approach to geometric algebra in section 3. Section 4 introduces the primitive geometric forms of synthetic projective geometry. Since coordinate systems are important tools in physics, we present in section 5 projective coordinate systems in different dimensions. Special emphasis is put on the dual construction of the projective coordinate systems. Section 6 is devoted to the incidence relations between the primitive geometric forms of section 4. The linear complex and the operations 'intersection' and 'connection' are discussed as well. On the base of the completely dual approach, section 7 provides the formulation of the principle of duality in the language of geometric algebra. The principle of duality is applied to Desargues' theorem.

2 Geometric Algebra

In this section we review some definitions and notations of geometric algebra. For a thorough introduction to the topic we refer to [HeSo].

A generic element of a geometric algebra is called a *multivector*. Throughout this article uppercase letters $A, B, \ldots$ and lowercase letters $a, b, \ldots$ denote multivectors and lowercase Greek letters $\alpha, \beta, \ldots$ denote real numbers. A *real geometric algebra* $\mathcal{G}_n$ is obtained by viewing the total of all multivectors as a linear space over the reals (with dimension 2^n) endowed with a *geometric product* defined by the following properties holding for all multivectors.

$$
\begin{aligned}
(AB)C &= A(BC) && \text{(1a)}\\
A(B+C) &= AB+AC && \text{(1b)}\\
(B+C)A &= BA+CA && \text{(1c)}\\
\mathbb{1}A &= A && \text{(1d)}\\
\lambda A &= A\lambda && \text{(1e)}
\end{aligned}
$$

(1a)–(1c) require the geometric algebra to be associative and distributive with respect to addition. (1d) guarantees a one-element for the geometric product, and (1e) expresses the commutativity of the scalar multiplication.

A real geometric algebra $\mathcal{G}_n$ is generated through continued geometric multiplication out of the vectors of a real n-dimensional vector space $\mathcal{V}$ ($\neq \mathcal{G}_n$). The *contraction rule* distinguishes the geometric algebra from a mere associative algebra by requiring the square of any vector of $\mathcal{V}$ to be a *scalar*,

$$A^2 = \lambda \mathbb{1} \qquad \forall\, A \in \mathcal{V}, \tag{2}$$

i.e., a multivector proportional to $\mathbb{1}$.

The geometric product of any two homogeneous multivectors $A_{\bar{r}}$ and $B_{\bar{s}}$ decomposes into a sum,

$$
\begin{aligned}
A_{\bar{r}}B_{\bar{s}} &= \langle A_{\bar{r}}B_{\bar{s}}\rangle_{|r-s|} + \langle A_{\bar{r}}B_{\bar{s}}\rangle_{|r-s|+2} + \ldots + \langle A_{\bar{r}}B_{\bar{s}}\rangle_{\mathcal{D}_n(r+s)}\\
&= \sum_{k=0}^{m} \langle A_{\bar{r}}B_{\bar{s}}\rangle_{|r-s|+2k},
\end{aligned} \tag{3}
$$

of k-vectors with $m := 2^{-1}(\mathcal{D}_n(r+s) - \mid r-s \mid)$ and the index function

$$\mathcal{D}_n(i) = \begin{cases} i & 0 \le i \le n \\ 2n - i & n \le i \le 2n. \end{cases} \tag{4}$$

From the geometric product we will define two more products. The *inner product*,

$$A_{\bar{r}} \cdot B_{\bar{s}} := \langle A_{\bar{r}}B_{\bar{s}}\rangle_{|r-s|}, \tag{5}$$

and the *outer product*,

$$A_{\bar{r}} \wedge B_{\bar{s}} := \langle A_{\bar{r}}B_{\bar{s}}\rangle_{r+s} \qquad r+s \le n, \tag{6a}$$

for any two homogeneous multivectors $A_{\bar{r}}$ and $B_{\bar{s}}$. In the case $r+s > n$ the outer product vanishes,

$$A_{\bar{r}} \wedge B_{\bar{s}} = 0 \qquad r+s > n, \tag{6b}$$

because the multivectors contain linearly dependent factors. In case of 1-vectors ($r = s = 1$), the geometric product decomposes into a sum of an inner and an outer product.

$$AB = A \cdot B + A \wedge B \qquad \forall\, A, B \in \mathcal{V} \tag{7}$$

	basis elements	direct subspace	dim.
scalar	$\mathbb{1}$	$\mathcal{G}^0_{p,q}$	$\binom{n}{0}$
vector	$P_1,\, P_2,\, P_3,\, \ldots\, ,\, P_n$	$\mathcal{G}^1_{p,q} \equiv \mathcal{V}$	$\binom{n}{1}$
bivector	$P_i P_j \quad \forall\, i < j$	$\mathcal{G}^2_{p,q}$	$\binom{n}{2}$
trivector	$P_i P_j P_k \quad \forall\, i < j < k$	$\mathcal{G}^3_{p,q}$	$\binom{n}{3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k-vector	$P_{i_1} P_{i_2} \cdots P_{i_k} \quad \forall\, i_1 < i_2 < \ldots < i_k$	$\mathcal{G}^k_{p,q}$	$\binom{n}{k}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(n-1)$-vector	$P_{i_1} P_{i_2} \cdots P_{i_{n-1}} \quad \forall\, i_1 < i_2 < \ldots < i_{n-1}$	$\mathcal{G}^{n-1}_{p,q}$	$\binom{n}{n-1}$
pseudoscalar	$\mathbb{I} := P_1 P_2 \cdots P_{n-1} P_n$	$\mathcal{G}^n_{p,q}$	$\binom{n}{n}$

Table 1: Basis of the geometric algebra $\mathcal{G}_{p,q}$.

The total of all k-vectors forms a linear space denoted by $\mathcal{G}^k_n$. In particular the one-dimensional linear space of all scalars is denoted by $\mathcal{G}^0_n$, the n-dimensional space of all vectors by $\mathcal{G}^1_n \equiv \mathcal{V}$, and the one-dimensional linear space of all pseudoscalars by $\mathcal{G}^n_n$.

To get the dimension of any linear subspace $\mathcal{G}^k_n$ we explicitly choose a basis in $\mathcal{G}^1_n$. For a non-degenerate geometric algebra there is always a basis P_i, $i = 1, \ldots, n = p + q$, where the inner products take the values

$$P_i \cdot P_j = \eta_{ij}\, \mathbb{1}, \tag{8}$$

with

$$\begin{array}{ll} \eta_{ij} = 0 & i \neq j \\ \eta_{ii} = 1 & 1 \leq i \leq p \\ \eta_{ii} = -1 & p+1 \leq i \leq p+q = n. \end{array} \tag{9}$$

The geometric algebra $\mathcal{G}_{p,q} \equiv \mathcal{G}_n$ is said to have a *signature* (p, q). Following the notation for the whole algebra the linear subspaces $\mathcal{G}^k_n$ are equivalently denoted by $\mathcal{G}^k_{p,q}$. A basis for the whole geometric algebra is shown in Table 1.

$M^\dagger$ denotes the *reverse* of the multivector M.

3 A Completely Dual Approach to Geometric Algebra

So far the approach to the geometric algebra $\mathcal{G}_{p,q}$ was based on the vectors P_i of the vector space $\mathcal{G}^1_{p,q}$, thus generating the whole algebra through continued geometric multiplication. From this development grew a space $\mathcal{G}_{p,q}$, the subspaces $\mathcal{G}_{p,q}^k$ of which are 'left-right' symmetrically distributed with regard to the dimensions.

$$\dim(\mathcal{G}_{p,q}^k) = \binom{n}{k} = \binom{n}{n-k} = \dim(\mathcal{G}_{p,q}^{n-k}) \tag{10}$$

In the light of this symmetry the vectors P_i lose their central role, and it is obvious that one would try to generate the *same* geometric algebra $\mathcal{G}_{p,q}$ from the $(n-1)$-vectors of the n-dimensional linear space $\mathcal{G}_{p,q}^{n-1}$ as well. This second approach will be implemented in this section.

3.1 Duality

The geometric product of any homogeneous multivector $A_{\bar{r}}$ with any pseudoscalar $N = \mu\mathbb{I}$ is a homogeneous multivector

$$B = \langle B \rangle_{n-r} = A_{\bar{r}} N = \mu A_{\bar{r}} \mathbb{I} \tag{11}$$

of grade $(n-r)$. Thus, geometric multiplication with a pseudoscalar mediates between the left-right symmetric subspaces $\mathcal{G}_{p,q}^r$ and $\mathcal{G}_{p,q}^{n-r}$. Following D. Hestenes and R. Ziegler [HeZi] we define the *dual* $\tilde{A}$ of a multivector A by geometric multiplication from the right side with the unit-n-blade $\mathbb{I}^{-1}$.

$$\tilde{A} := A\mathbb{I}^{-1} = A \cdot \mathbb{I}^{-1} \tag{12}$$

Table 2 lists frequently used properties of the unit pseudoscalar $\mathbb{I}$.

$\mathbb{I}$	$:=$	$P_1 P_2 \cdots P_{n-1} P_n$	$\mathbb{I}^{-1}$	$=$	$(-1)^{\frac{n(n-1)}{2}}(-1)^q \mathbb{I}$
$\mathbb{I}^\dagger$	$=$	$(-1)^{\frac{n(n-1)}{2}} \mathbb{I}$	$\mathbb{I}^2$	$=$	$[\mathbb{I}^{-1}]^2 = [\mathbb{I}^\dagger]^2 = (-1)^{\frac{n(n-1)}{2}}(-1)^q \mathbb{1}$
$P_i \mathbb{I}$	$=$	$(-1)^{n-1} \mathbb{I} P_i$	$A_{\bar{k}} \mathbb{I}$	$=$	$(-1)^{k(n-1)} \mathbb{I} A_{\bar{k}}$

Table 2: Properties of the unit pseudoscalar $\mathbb{I}$.

3.2 Dual Geometric Product

With a commutative diagram of mappings we introduce a new $*$-*product*,

$$\begin{array}{ccc} \mathcal{G}_{p,q} \otimes \mathcal{G}_{p,q} & \text{geometric product} & \mathcal{G}_{p,q} \\ (A, B) & \longrightarrow & AB \\ \downarrow \mathbb{I}^{-1} & & \downarrow \mathbb{I}^{-1} \\ \mathcal{G}_{p,q} \otimes \mathcal{G}_{p,q} & *\text{-product} & \mathcal{G}_{p,q} \\ (\tilde{A}, \tilde{B}) & \longrightarrow & \tilde{A} * \tilde{B} := (AB)^{\sim} \end{array} \tag{13}$$

basis elements	**direct subspace**	**dim.**
$\mathbb{1}^- := \tilde{\mathbb{1}} = \mathbb{I}^{-1}$	$\mathcal{G}^{0-}_{p,q} \equiv \mathcal{G}^{n}_{p,q}$	$\binom{n}{0}$
$P_i^- := \tilde{P}_i = P_i\mathbb{I}^{-1} \qquad \forall\, i \in \{1, 2, \ldots, n\}$	$\mathcal{G}^{1-}_{p,q} \equiv \mathcal{G}^{n-1}_{p,q}$	$\binom{n}{1}$
$\quad = \mathbb{I}^2(-1)^{i-1}\eta_{ii}P_1 \cdots P_{i-1}P_{i+1} \cdots P_n$		
$P_i^- * P_j^- = \tilde{P}_i * \tilde{P}_j = P_iP_j\mathbb{I}^{-1} \qquad \forall\, i < j$	$\mathcal{G}^{2-}_{p,q} \equiv \mathcal{G}^{n-2}_{p,q}$	$\binom{n}{2}$
$\quad = \mathbb{I}^2(-1)^{(i+j)-2}\eta_{ii}\eta_{jj}P_1 \cdots P_{i-1}P_{i+1} \cdots P_{j-1}P_{j+1} \cdots P_n$		
$P_i^- * P_j^- * P_k^- = \qquad \forall\, i < j < k$	$\mathcal{G}^{3-}_{p,q} \equiv \mathcal{G}^{n-3}_{p,q}$	$\binom{n}{3}$
$\quad = \tilde{P}_i * \tilde{P}_j * \tilde{P}_k$		
$\quad = (P_iP_j\mathbb{I}^{-1}) * (P_k\mathbb{I}^{-1})$		
$\quad = P_iP_jP_k\mathbb{I}^{-1}$		
$\quad = \mathbb{I}^2(-1)^{(i+j+k)-3}\eta_{ii}\eta_{jj}\eta_{kk}P_1 \cdots P_{i-1}P_{i+1} \cdots P_{j-1}P_{j+1} \cdots P_{k-1}P_{k+1} \cdots P_n$		
$\vdots$	$\vdots$	$\vdots$
$P_{i_1}^- * P_{i_2}^- * \cdots * P_{i_k}^- = \qquad \forall\, i_1 < i_2 < \ldots < i_k$	$\mathcal{G}^{k-}_{p,q} \equiv \mathcal{G}^{n-k}_{p,q}$	$\binom{n}{k}$
$\quad = \tilde{P}_{i_1} * \tilde{P}_{i_2} * \cdots * \tilde{P}_{i_{k-1}} * \tilde{P}_{i_k}$		
$\quad = (P_{i_1}P_{i_2} \cdots P_{i_{k-1}}\mathbb{I}^{-1}) * (P_{i_k}\mathbb{I}^{-1})$		
$\quad = P_{i_1}P_{i_2} \cdots P_{i_{k-1}}P_{i_k}\mathbb{I}^{-1}$		
$\quad = \mathbb{I}^2(-1)^{\sum_{l=1}^{k}(i_l-1)}(\prod_{l=1}^{k}\eta_{i_li_l})P_1 \cdots P_{i_1-1}P_{i_1+1} \cdots\cdots P_{i_k-1}P_{i_k+1} \cdots P_n$		
$\vdots$	$\vdots$	$\vdots$
$P_{i_1}^- * P_{i_2}^- * \cdots * P_{i_{n-1}}^- = \qquad \forall\, i_1 < i_2 < \ldots < i_{n-1}$	$\mathcal{G}^{(n-1)-}_{p,q} \equiv \mathcal{G}^{1}_{p,q}$	$\binom{n}{n-1}$
$\quad = \tilde{P}_{i_1} * \tilde{P}_{i_2} * \cdots * \tilde{P}_{i_{n-1}}$		
$\quad = P_{i_1}P_{i_2} \cdots P_{i_{n-2}}P_{i_{n-1}}\mathbb{I}^{-1}$		
$\mathbb{I}^- := P_1^- * P_2^- * \cdots * P_{n-1}^- * P_n^-$	$\mathcal{G}^{n-}_{p,q} \equiv \mathcal{G}^{0}_{p,q}$	$\binom{n}{n}$
$\quad = P_1P_2 \cdots P_{n-1}P_n\mathbb{I}^{-1}$		
$\quad = \mathbb{1}$		

Table 3: Basis of the geometric algebra $\mathcal{G}^-_{p,q}$. The new 'minus' basis elements are translated into the basis of Table 1.

which translates into the given geometric product according to

$$
\begin{aligned}
\tilde{A} * \tilde{B} &= (AB)^\sim, && \text{(14a)}\\
A * B &= (\tilde{A}\tilde{B})^\sim. && \text{(14b)}
\end{aligned}
$$

The $*$-product is associative,

$$
\begin{aligned}
(A * B) * C &= (\tilde{A}\tilde{B})^\sim * C = [\mathbb{I}^2(\tilde{A}\tilde{B})\tilde{C}]^\sim \\
&= [\mathbb{I}^2\tilde{A}(\tilde{B}\tilde{C})]^\sim = A * (\tilde{B}\tilde{C})^\sim \qquad (15)\\
&= A * (B * C),
\end{aligned}
$$

distributive with respect to addition,

$$\begin{aligned} A * (B + C) &= [\tilde{A}(\tilde{B} + \tilde{C})]^{\sim} = (\tilde{A}\tilde{B})^{\sim} + (\tilde{A}\tilde{C})^{\sim} \\ &= A * B + A * C, \end{aligned} \tag{16a}$$

$$\begin{aligned} (B + C) * A &= [(\tilde{B} + \tilde{C})\tilde{A}]^{\sim} = (\tilde{B}\tilde{A})^{\sim} + (\tilde{C}\tilde{A})^{\sim} \\ &= B * A + C * A, \end{aligned} \tag{16b}$$

and the pseudoscalar $\tilde{\mathbb{I}} = \mathbb{I}^{-1}$ plays the role of the one-element,

$$\tilde{\mathbb{I}} * A = \mathbb{I}^{-1} * A = \mathbb{I}^4 A = A. \tag{17}$$

In addition, there is a dual contraction rule with respect to (2). The square of any $(n-1)$-vector A is a pseudoscalar,

$$A * A = \tilde{A}^2 \mathbb{I}^{-1} = \lambda \mathbb{I}^{-1} = \lambda \tilde{\mathbb{I}}, \tag{18}$$

i.e., a multivector proportional to the one-element of the $*$-product. Thus, the $*$-product satisfies the axioms (1) and (2) and represents a new geometric product.

The original geometric product and the *dual geometric product* $*$ act in the same geometric algebra $\mathcal{G}_{p,q}$ (viewed as a linear space). The first one generates the geometric algebra from the vectors of the vector space $\mathcal{G}^1_{p,q}$, and the second one does so from the $(n-1)$-vectors of the linear space $\mathcal{G}^{n-1}_{p,q}$; see Table 3. Thus, we have shown how to generate the same geometric algebra by continued dual geometric multiplication starting with the homogeneous multivectors of the n-dimensional linear space $\mathcal{G}^{n-1}_{p,q}$.

3.3 One Space with Two Aspects—Two Spaces for One Aspect

Left-right symmetry demands another consequence. If the geometric algebra $\mathcal{G}_{p,q}$ is developed from the vectors P_i^- (Table 3) as well as from the vectors P_i (Table 1) it should also be possible to adjust the grade of the homogeneous multivectors to the actual approach in the first case. For this reason we introduce a plus-minus notation. It will distinguish between the left and the right approach to the geometric algebra. From now on we supply the vectors of the vector space $\mathcal{G}^1_{p,q}$ with an extra plus sign, $A^+_{\bar{1}} \equiv A_{\bar{1}}$, as well as the vector space itself, $\mathcal{G}^{1+}_{p,q} \equiv \mathcal{G}^1_{p,q}$. The same applies to any k-vector $A^+_{\bar{k}} \equiv A_{\bar{k}}$, multivector $A^+ \equiv A$, subspace $\mathcal{G}^{k+}_{p,q} \equiv \mathcal{G}^k_{p,q}$, and the geometric algebra $\mathcal{G}^+_{p,q} \equiv \mathcal{G}_{p,q}$. A plus sign indicates that the geometric algebra $\mathcal{G}^+_{p,q}$ is generated by the vectors P_i^+ of the vector space $\mathcal{G}^{1+}_{p,q}$. Equivalently, a minus sign points out that the geometric algebra $\mathcal{G}^-_{p,q}$ is generated from the vectors P_i^- of the vector space $\mathcal{G}^{1-}_{p,q}$. If the geometric algebras $\mathcal{G}^+_{p,q}$ and $\mathcal{G}^-_{p,q}$ are considered as *linear spaces*, there is no difference between them, i.e., we may speak of *one space*,

$$\mathcal{G}^+_{p,q} \equiv \mathcal{G}_{p,q} \equiv \mathcal{G}^-_{p,q}. \tag{19}$$

The linear space $\mathcal{G}_{p,q}$ is endowed with two geometric products denoted by juxtaposition and $*$. If the geometric product denoted by the juxtaposition is taken as the original geometric product, the linear space $\mathcal{G}_{p,q}$ turns into the geometric algebra $\mathcal{G}^+_{p,q}$. If the $*$-product represents the original geometric product, $\mathcal{G}_{p,q}$ turns into the geometric algebra $\mathcal{G}^-_{p,q}$. Thus, the geometric algebras $\mathcal{G}^+_{p,q}$ and $\mathcal{G}^-_{p,q}$ represent *two* different *aspects* of the same linear space $\mathcal{G}_{p,q}$.

By virtue of the isomorphisms

$$\begin{aligned} \psi^+ : \quad \mathcal{G}^-_{p,q} &\longrightarrow \mathcal{G}^+_{p,q} \\ A^-_{\bar{k}} &\longmapsto \langle A^-_{\bar{k}} \rangle^+_{n-k}, \end{aligned} \tag{20a}$$

$$\begin{aligned} \psi^- : \quad \mathcal{G}^+_{p,q} &\longrightarrow \mathcal{G}^-_{p,q} \\ A^+_{\bar{k}} &\longmapsto \langle A^+_{\bar{k}} \rangle^-_{n-k}, \end{aligned} \tag{20b}$$

the geometric algebra $\mathcal{G}^+_{p,q}$ contains $\mathcal{G}^-_{p,q}$ (20a) and the geometric algebra $\mathcal{G}^-_{p,q}$ contains $\mathcal{G}^+_{p,q}$ (20b). Each basis element, homogeneous multivector, and subspace of the linear space $\mathcal{G}_{p,q}$ appears in two different grades.

$$\langle P^-_i \rangle^+_{n-1} \equiv P^-_i \tag{21a}$$

$$P^+_i \equiv \langle P^+_i \rangle^-_{n-1} \tag{21b}$$

$$A^+_{\bar{k}} \equiv \langle A^+_{\bar{k}} \rangle^-_{n-k} \tag{22a}$$

$$\langle A^-_{\bar{k}} \rangle^+_{n-k} \equiv A^-_{\bar{k}} \tag{22b}$$

$$\mathcal{G}^{k+}_{p,q} \equiv \mathcal{G}^{(n-k)-}_{p,q} \tag{23a}$$

$$\mathcal{G}^{(n-k)+}_{p,q} \equiv \mathcal{G}^{k-}_{p,q} \tag{23b}$$

On the other hand we may stress the fact that the geometric algebras $\mathcal{G}^+_{p,q}$ and $\mathcal{G}^-_{p,q}$ are equal regarding their form and their construction, i.e., we may speak of *one aspect*,

$$\mathcal{G}^+_{p,q} \simeq \mathcal{G}^-_{p,q}. \tag{24}$$

Thus, the structure of geometric algebra is realized in *two spaces* $\mathcal{G}^+_{p,q}$ and $\mathcal{G}^-_{p,q}$.

By virtue of the isomorphisms

$$\begin{aligned} \phi^+ : \quad \mathcal{G}^-_{p,q} &\longrightarrow \mathcal{G}^+_{p,q} \\ A^- &\longmapsto \langle A^- * (\mathbb{I}^-)^{-1} \rangle^+, \end{aligned} \tag{25a}$$

$$\begin{aligned} \phi^- : \quad \mathcal{G}^+_{p,q} &\longrightarrow \mathcal{G}^-_{p,q} \\ A^+ &\longmapsto \langle A^+ (\mathbb{I}^+)^{-1} \rangle^-, \end{aligned} \tag{25b}$$

each grade is represented by two different homogeneous multivectors and subspaces.

$$A_{\bar{k}}^{+} \simeq \langle A_{\bar{k}}^{+}(\mathbb{I}^{+})^{-1}\rangle_{k}^{-} \tag{26a}$$

$$\langle A_{\bar{k}}^{-} * (\mathbb{I}^{-})^{-1}\rangle_{k}^{+} \simeq A_{\bar{k}}^{-} \tag{26b}$$

$$\mathcal{G}_{p,q}^{k+} \simeq \mathcal{G}_{p,q}^{k-} \tag{27}$$

We will not always apply the plus-minus notation rigorously. If plus and minus signs are absent, the explanations and statements hold for *both* signs. For instance, the basis elements and subspaces of Table 1 can be supplied with a plus as well as with a minus sign. Both cases are correct. All the concepts introduced so far hold rigorously for both aspects.

Explicit computation of the square and the inverse of the pseudoscalar $\mathbb{I}^{-}$,

$$(\mathbb{I}^{-})^{2} = \mathbb{I}^{-} * \mathbb{I}^{-} = (-1)^{\frac{n(n-1)}{2}}(-1)^{q}\mathbb{1}^{-}, \tag{28}$$

$$(\mathbb{I}^{-})^{-1} = (-1)^{\frac{n(n-1)}{2}}(-1)^{q}\mathbb{I}^{-}, \tag{29}$$

confirm the equivalence to the corresponding formulas in Table 2. Duality in the geometric algebra $\mathcal{G}_{p,q}^{-}$ complies with the definition (12) in the geometric algebra $\mathcal{G}_{p,q}^{+}$,

$$A * (\mathbb{I}^{-})^{-1} = \tilde{A}, \tag{30}$$

and for the translation of the original geometric product in $\mathcal{G}_{p,q}^{+}$ into the original geometric product in $\mathcal{G}_{p,q}^{-}$, we obtain

$$\tilde{A}\tilde{B} = (\mathbb{I}^{-})^{2}(A * B)^{\sim}, \tag{31a}$$

$$AB = (\mathbb{I}^{-})^{2}(\tilde{A} * \tilde{B})^{\sim}, \tag{31b}$$

thus providing the inverse relations of (14).

3.4 Inner and Outer Products

In the geometric algebra $\mathcal{G}_{p,q}^{+}$ as well as in the geometric algebra $\mathcal{G}_{p,q}^{-}$ there is a geometric product, an outer product and an inner product. To distinguish between these *six* different products we introduce, in addition to the geometric product, a new notation for the inner and outer products in the geometric algebra $\mathcal{G}_{p,q}^{-}$; see Table 4. According to (5) and (6) the inner and outer products in the geometric algebra $\mathcal{G}_{p,q}^{-}$ are defined by

$$A_{\bar{r}}^{-} \circ B_{\bar{s}}^{-} = \langle A_{\bar{r}}^{-} * B_{\bar{s}}^{-}\rangle_{|r-s|}^{-}, \tag{32}$$

$$A_{\bar{r}}^{-} \vee B_{\bar{s}}^{-} = \langle A_{\bar{r}}^{-} * B_{\bar{s}}^{-}\rangle_{r+s}^{-} \qquad r+s \leq n, \tag{33a}$$

	geometric product	inner product	outer product
$\mathcal{G}^+_{p,q}$	(juxtaposition)	$\cdot$	$\wedge$
$\mathcal{G}^-_{p,q}$	$*$	$\circ$	$\vee$

Table 4: Notation of the products in $\mathcal{G}^+_{p,q}$ and $\mathcal{G}^-_{p,q}$.

with

$$A^-_{\bar{r}} \vee B^-_{\bar{s}} = 0 \qquad r+s>n, \tag{33b}$$

because the homogeneous multivectors $A^-_{\bar{r}}$ and $B^-_{\bar{s}}$ contain linearly dependent factors in this case. Dual to (3), the geometric product $*$ between two homogeneous multivectors is decomposed into a sum of homogeneous multivectors,

$$\begin{aligned} A^-_{\bar{r}} * B^-_{\bar{s}} &= \langle A^-_{\bar{r}} * B^-_{\bar{s}} \rangle^-_{|r-s|} + \langle A^-_{\bar{r}} * B^-_{\bar{s}} \rangle^-_{|r-s|+2} + \dots \\ &\quad \dots + \langle A^-_{\bar{r}} * B^-_{\bar{s}} \rangle^-_{\mathcal{D}_n(r+s)} \\ &= \sum_{k=0}^{m} \langle A^-_{\bar{r}} * B^-_{\bar{s}} \rangle^-_{|r-s|+2k}, \end{aligned} \tag{34}$$

with $m := 2^{-1}(\mathcal{D}_n(r+s) - \mid r-s \mid)$ and the index function (4).
In the case $r=s=1$ we obtain the dual expression to (7).

$$A^-_{\bar{1}} * B^-_{\bar{1}} = A^-_{\bar{1}} \circ B^-_{\bar{1}} + A^-_{\bar{1}} \vee B^-_{\bar{1}} \tag{35}$$

The inner products in $\mathcal{G}^-_{p,q}$ between two basis elements P^-_i,

$$\begin{aligned} P^-_i \circ P^-_j &= \langle P^-_i * P^-_j \rangle^-_0 = \langle P^-_i (\mathbb{I}^+)^{-1} P^-_j (\mathbb{I}^+)^{-1} (\mathbb{I}^+)^{-1} \rangle^+_n \\ &= \langle P^+_i P^+_j \rangle^+_0 (\mathbb{I}^+)^{-1} = P^+_i \cdot P^+_j (\mathbb{I}^+)^{-1} \\ &= \eta_{ij} \mathbb{1}^-, \end{aligned} \tag{36}$$

are in exact correspondence to (8).

From the translation law for the geometric products (14) we get the duality of the outer products of $\mathcal{G}^+_{p,q}$ and $\mathcal{G}^-_{p,q}$,

$$\begin{aligned} (A^+_{\bar{r}})^\sim \vee (B^+_{\bar{s}})^\sim &= \langle (A^+_{\bar{r}})^\sim \rangle^-_r \vee \langle (B^+_{\bar{s}})^\sim \rangle^-_s = \langle (A^+_{\bar{r}})^\sim * (B^+_{\bar{s}})^\sim \rangle^-_{r+s} \\ &= \langle A^+_{\bar{r}} B^+_{\bar{s}} \mathbb{I}^{-1} \rangle^+_{n-(r+s)} = \langle A^+_{\bar{r}} B^+_{\bar{s}} \rangle^+_{r+s} \mathbb{I}^{-1} \\ &= (A^+_{\bar{r}} \wedge B^+_{\bar{s}})^\sim, \end{aligned} \tag{37}$$

with $r+s \leq n$. For the complementary grades $r+s \geq n$ the equation (37) holds trivially. Thus, for generic multivectors A and B we have

$$\tilde{A} \vee \tilde{B} = (A \wedge B)^\sim, \tag{38a}$$

$$A \vee B = (\tilde{A} \wedge \tilde{B})^\sim, \tag{38b}$$

	(juxtaposition)	$*$	$\cdot$	$\circ$	$\wedge$	$\vee$
$\mathcal{G}^+_{p,q}$	original	dual	original	dual	progressive	regressive
$\mathcal{G}^-_{p,q}$	dual	original	dual	original	regressive	progressive

Table 5: Specifying terms for geometric, inner, and outer products depending on the chosen aspect.

and equivalently from the translation law (31)

$$\tilde{A} \wedge \tilde{B} = (\mathbb{I}^-)^2 (A \vee B)^\sim, \tag{39a}$$

$$A \wedge B = (\mathbb{I}^-)^2 (\tilde{A} \vee \tilde{B})^\sim. \tag{39b}$$

The relation (38) is a natural consequence of the completely dual approach and expresses the duality of the outer products $\wedge$ and $\vee$. In the usual one-sided construction of the geometric algebra, the relation (38) is taken as the definition for the product $\vee$ and one may then show that it also has the structure of an outer product, see for example [HeSo, HeZi]. The latter step is unnecessary in our approach, thus already indicating the power of the dual approach.

Another new result is the duality relation between the inner products of $\mathcal{G}^+_{p,q}$ and $\mathcal{G}^-_{p,q}$.

$$\begin{aligned}
(A^+_{\bar{r}})^\sim \circ (B^+_{\bar{s}})^\sim &= \langle (A^+_{\bar{r}})^\sim \rangle^-_r \circ \langle (B^+_{\bar{s}})^\sim \rangle^-_s = \langle (A^+_{\bar{r}})^\sim * (B^+_{\bar{s}})^\sim \rangle^-_{|r-s|} \\
&= \langle A^+_{\bar{r}} B^+_{\bar{s}} \mathbb{I}^{-1} \rangle^+_{n-|r-s|} = \langle A^+_{\bar{r}} B^+_{\bar{s}} \rangle^+_{|r-s|} \mathbb{I}^{-1} \\
&= (A^+_{\bar{r}} \cdot B^+_{\bar{s}})^\sim
\end{aligned} \tag{40}$$

Thus, for generic multivectors A and B we have

$$\tilde{A} \circ \tilde{B} = (A \cdot B)^\sim, \tag{41a}$$

$$A \circ B = (\tilde{A} \cdot \tilde{B})^\sim, \tag{41b}$$

and equivalently

$$\tilde{A} \cdot \tilde{B} = (\mathbb{I}^-)^2 (A \circ B)^\sim, \tag{42a}$$

$$A \cdot B = (\mathbb{I}^-)^2 (\tilde{A} \circ \tilde{B})^\sim. \tag{42b}$$

The inner and outer products of $\mathcal{G}^+_{p,q}$ and separately of $\mathcal{G}^-_{p,q}$ are connected by the *dual conjugation*.

$$
\begin{aligned}
(A^+_{\bar{r}} \wedge B^+_{\bar{s}})^\sim &= \langle A^+_{\bar{r}} B^+_{\bar{s}} \rangle_{r+s} \mathbb{I}^{-1} \\
&= \langle A^+_{\bar{r}} B^+_{\bar{s}} \mathbb{I}^{-1} \rangle_{n-(r+s)} = (-1)^{s(n-1)} \langle A^+_{\bar{r}} \mathbb{I}^{-1} B^+_{\bar{s}} \rangle_{n-(r+s)} \\
&= \begin{cases} A^+_{\bar{r}} \cdot (B^+_{\bar{s}})^\sim & r+s \leq n \\ (-1)^{s(n-1)} (A^+_{\bar{r}})^\sim \cdot B^+_{\bar{s}} & r+s \leq n \end{cases} \qquad (43) \\
(A^-_{\bar{r}} \vee B^-_{\bar{s}})^\sim &= \begin{cases} A^-_{\bar{r}} \circ (B^-_{\bar{s}})^\sim & r+s \leq n \\ (-1)^{s(n-1)} (A^-_{\bar{r}})^\sim \circ B^-_{\bar{s}} & r+s \leq n \end{cases} \qquad (44) \\
(A^+_{\bar{r}} \cdot B^+_{\bar{s}})^\sim &= \langle A^+_{\bar{r}} B^+_{\bar{s}} \rangle_{|r-s|} \mathbb{I}^{-1} \\
&= \langle A^+_{\bar{r}} B^+_{\bar{s}} \mathbb{I}^{-1} \rangle_{n-|r-s|} = (-1)^{s(n-1)} \langle A^+_{\bar{r}} \mathbb{I}^{-1} B^+_{\bar{s}} \rangle_{n-|r-s|} \\
&= \begin{cases} A^+_{\bar{r}} \wedge (B^+_{\bar{s}})^\sim & r \leq s \\ (-1)^{s(n-1)} (A^+_{\bar{r}})^\sim \wedge B^+_{\bar{s}} & r \geq s \end{cases} \qquad (45) \\
(A^-_{\bar{r}} \circ B^-_{\bar{s}})^\sim &= \begin{cases} A^-_{\bar{r}} \vee (B^-_{\bar{s}})^\sim & r \leq s \\ (-1)^{s(n-1)} (A^-_{\bar{r}})^\sim \vee B^-_{\bar{s}} & r \geq s \end{cases} \qquad (46)
\end{aligned}
$$

In the geometric algebra $\mathcal{G}^+_{p,q}$ the outer product $\wedge$ between two homogeneous multivectors $A^+_{\bar{r}}$ and $B^+_{\bar{s}}$ is a vector $A^+_{\bar{r}} \wedge B^+_{\bar{s}}$ of grade $r+s$ $(r, s \leq r+s \leq n)$. It is therefore said to be *progressive*. The outer product $A^+_{\bar{r}} \vee B^+_{\bar{s}}$ forms a homogeneous multivector of grade $r+s-n$ $(r+s-n \leq r, s \leq n)$ and is thus *regressive*. From the point of view of the geometric algebra $\mathcal{G}^-_{p,q}$ the outer product $\vee$ is progressive and $\wedge$ is regressive. The specifying terms for the six different products are compiled in Table 5.

4 Primitive Geometric Forms

There are three different elements in the geometry of space: *point*, *line* and *plane*. Synthetic projective geometry treats these elements equally, i.e., on an elementary level point, line, and plane are thought of as *pure* elements. A pure point is thought to exist without lines and planes passing through it, a pure line is thought to exist without points lying in it and planes passing through it, and a pure plane is thought to exist without lines and points lying in it. Thus, a pure element doesn't enter into an incidence relation to either of the two other elements. The equal handling of the elements has far reaching consequences for geometry. A first one is that the set of all pure planes, the three dimensional *space of planes*, opens up an alternative illustration for a three dimensional manifold besides the three dimensional *space of points*, the set of all pure points. The set of all pure lines, the *space of lines*, forms a four dimensional manifold.

Having rejected all the incidence relations between the elements, we re-introduce them again systematically. There are infinitely many points lying in and infinitely many planes passing through a line. The line appears under the aspect 'point' as a *pencil of points* (the set of all points lying in the line) and under the aspect 'plane' as a *pencil of planes* (the set of all planes passing through the line). There are infinitely many lines and infinitely many points

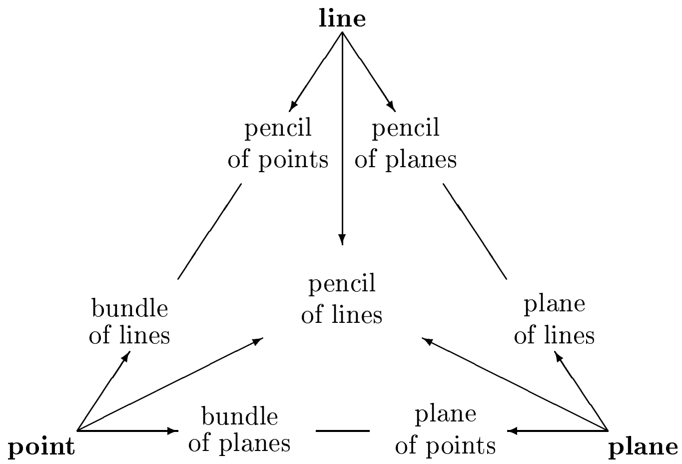

Figure 1: The incidence relations between the pure elements.

dimension	primitive geometric form	proj. space
-	point, line, plane	-
1	pencil of points, pencil of lines, pencil of planes	$\mathbb{P}^1$
2	{ planar field: plane of points, plane of lines centric bundle: bundle of planes, bundle of lines }	$\mathbb{P}^2$
3	space: space of points, space of planes	$\mathbb{P}^3$
4	space of lines	-

Table 6: Primitive geometric forms of synthetic projective geometry.

lying in a plane. The plane appears under the aspect 'line' as a *plane of lines* (the set of all lines lying in the plane) and under the aspect 'point' as a *plane of points* (the set of all points lying in the plane). There are infinitely many lines and infinitely many planes passing through a point. The point appears under the aspect 'line' as a *bundle of lines* (the set of all lines passing through the point) and under the aspect 'plane' as a *bundle of planes* (the set of all planes passing through the point). There are infinitely many lines lying in the plane and passing through the point of a incident plane-point pair. The plane-point pair appears under the aspect 'line' as a *pencil of lines* (the set of all lines which are at once on the same point and the same plane). A complete overview over the incidence relations between the pure elements is given in the diagram of Figure 1. The seven incidence relations, together with the space of planes, the space of points, and the space of lines, constitute the ten *primitive geometric forms* listed in Table 6 with their dimensions.

$\mathcal{G}_2^+$	**pencil of points**	**pencil of lines**	**pencil of planes**	$\mathcal{G}_2^-$
$\rho\mathbb{1}^+$				$\rho\mathbb{I}^-$
$\left.\begin{array}{r}\rho A_{\bar{1}}^+ = \mu_1^+ P_1^+ + \\ + \mu_2^+ P_2^+\end{array}\right\}$	point	line	plane	$\left\{\begin{array}{r}\rho A_{\bar{1}}^- = \mu_1^- P_1^- + \\ + \mu_2^- P_2^-\end{array}\right.$
$\rho\mathbb{I}^+$				$\rho\mathbb{1}^-$

Table 7: The geometric interpretation for a 1-vector of $\mathcal{G}_2^+$ and $\mathcal{G}_2^-$ depends on the chosen type of pencil.

5 The Order of Pure Elements in Primitive Geometric Forms. Projective Coordinate Systems

The order of pure elements can be treated within synthetic projective geometry as shown by O. Veblen and J. W. Young (chapter I and II of volume II in [VeYo]) and by L. Locher (chapter I in [Lo40] and p. 37ff of [Lo57]). We will follow another direction and establish one-to-one correspondences between primitive geometric forms and geometric algebras. These correspondences lead to projective coordinate systems. The order of the pure elements then is reflected by the order of the homogeneous multivectors. We will not consider the order relations in geometric algebra here. This section aims only at developing projective coordinate systems in primitive geometric forms. As a natural product from the introduction of a projective coordinate system in space, the linear complex, a generalized line in space, will be included into the set of geometric objects illustrating homogeneous multivectors.

The projective interpretation for homogeneous multivectors developed by D. Hestenes and R. Ziegler in section 3 of [HeZi] is unique to within a scale factor. Thus, a real geometric algebra $\mathcal{G}_{n+1}$ of dimension 2^{n+1} contains the real projective space $\mathbb{P}^n$. We take this as a prerequisite to our introduction of projective coordinate systems. Special emphasis is put on duality, thereby yielding two dual coordinate systems in every dimension.

Before we begin a detailed development, a warning is in order: A projective coordinate system *does not determine the signature* (or the metric) in the corresponding primitive geometric form. It *only reflects the order* of the pure elements. The choice of signature remains still open after introducing a projective coordinate system.

5.1 Pencils

The pencil of points, the pencil of lines, and the pencil of planes are three different examples for an one-dimensional projective space. Each of them is represented algebraically by the four-($= 2^2$)-dimensional geometric algebra $\mathcal{G}_2$. The homogeneous multivectors, together with their geometric interpretation, are shown in Table 7. According to Table 3, the translation of the basis elements of

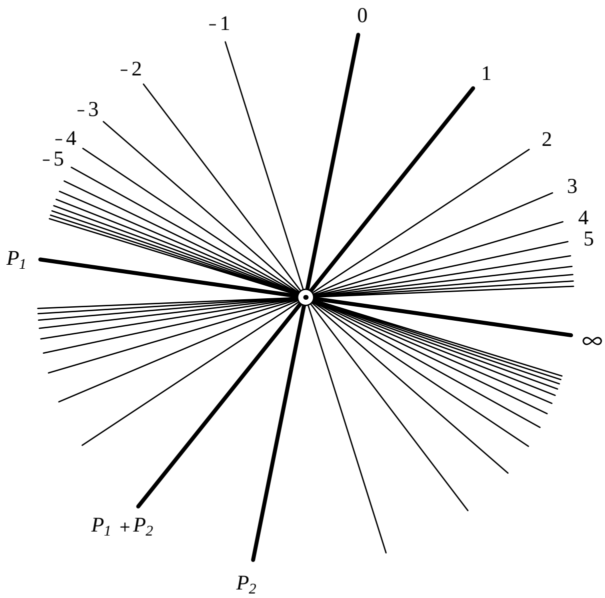

Figure 2: Projective line-coordinates of a pencil of lines.

the geometric algebra $\mathcal{G}_2^+$ into the basis elements of the geometric algebra $\mathcal{G}_2^-$ is given by

$$\mathbb{1}^+ = \mathbb{I}^-, \tag{47a}$$
$$\{\eta_{11}P_1^+, \eta_{22}P_2^+\} = \{P_2^-, -P_1^-\}, \tag{47b}$$
$$\eta_{11}\eta_{22}\mathbb{I}^+ = -\mathbb{1}^-. \tag{47c}$$

A projective coordinate system in a pencil is a continuous one-to-one correspondence between the elements of the pencil and the 1-vectors of the geometric algebra $\mathcal{G}_2$. It is well defined by any choice of the fundamental elements 0, 1, and ∞; see, for example, chapter V in [Kl] or chapter VI of volume I in [VeYo]. We may choose any three elements and assign to them the *homogeneous* coordinates $\rho(0,1)$, $\rho(1,1)$, and $\rho(1,0)$ in any order. Thus, the fundamental element $\rho(0,1)$ is represented by the second basis vector ρP_2, the fundamental element $\rho(1,1)$ by the 1-vector $\rho(P_1 + P_2)$, and the fundamental element $\rho(1,0)$ by the first basis vector ρP_1. Any 1-vector $\rho A_{\bar{1}} = \mu_1 P_1 + \mu_2 P_2$ belongs to the element with homogeneous coordinates $\rho(\mu_1, \mu_2)$ or the projective coordinate $\frac{\mu_1}{\mu_2}$.

This manner of introducing a coordinate system applies in both approaches to the geometric algebra $\mathcal{G}_2$. In the first case the 1-vectors and coordinates are supplied with plus signs, in the second case with minus signs. The connection between these dual coordinate systems is given by the equations (47). From

$$\mu_1^+ P_1^+ + \mu_2^+ P_2^+ = \mu_1^- P_1^- + \mu_2^- P_2^- \tag{48}$$

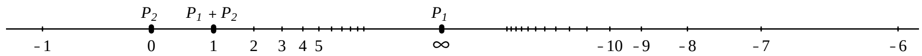

Figure 3: Projective point-coordinates of a pencil of points. The fundamental points 0, 1, and ∞ are arbitrarily chosen. The remaining projective coordinates are determined by the logarithm of the cross-ratio; see [Kl].

we obtain

$$\mu^- := \frac{\mu_1^-}{\mu_2^-} = -\left(\frac{\eta_{22}}{\eta_{11}}\right)\frac{1}{\frac{\mu_1^+}{\mu_2^+}} =: -\left(\frac{\eta_{22}}{\eta_{11}}\right)\frac{1}{\mu^+}. \tag{49}$$

Thus, the projective coordinate of a fixed element in one coordinate system equals the positive or negative reciprocal of the coordinate in the dual coordinate system. The sign of the reciprocal depends on the signature of the geometric algebra $\mathcal{G}_2$. It is positive for $\eta_{11} = -\eta_{22}$ and negative for $\eta_{11} = \eta_{22}$.

5.1.1 Pencil of Lines

In a pencil of lines, a generic multivector M of $\mathcal{G}_2$ decomposes into two different types of numbers and a line.

$$M = \langle\text{scalar}\rangle_0 + \langle\text{line}\rangle_1 + \langle\text{pseudoscalar}\rangle_2 \tag{50}$$

Figure 2 shows an example for the coordinatization of a pencil of lines.

5.1.2 Pencil of Points

In a pencil of points, a generic multivector M of $\mathcal{G}_2$ decomposes into two different types of numbers and a point.

$$M = \langle\text{scalar}\rangle_0 + \langle\text{point}\rangle_1 + \langle\text{pseudoscalar}\rangle_2. \tag{51}$$

Figure 3 shows an example for the coordinatization of a pencil of points.

5.1.3 Pencil of Planes

In a pencil of planes, a generic multivector M of $\mathcal{G}_2$ decomposes into two different types of numbers and a plane.

$$M = \langle\text{scalar}\rangle_0 + \langle\text{plane}\rangle_1 + \langle\text{pseudoscalar}\rangle_2 \tag{52}$$

Figure 4 shows an example for the coordinatization of a pencil of planes.

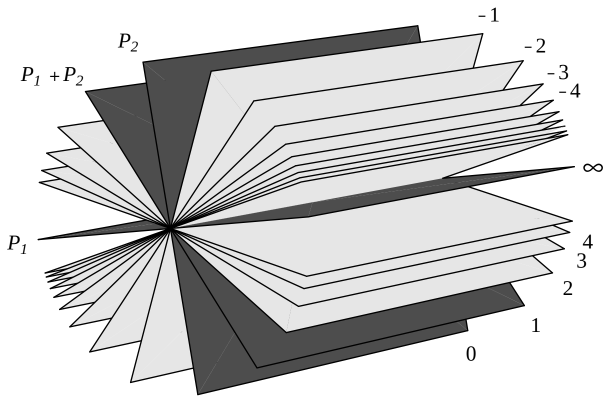

Figure 4: Projective plane-coordinates of a pencil of planes.

5.2 Bundle and Field

The plane of points, the plane of lines, the bundle of lines, and the bundle of planes are four different examples of two-dimensional projective spaces $\mathbb{P}^2$. The planar field and the centric bundle are represented algebraically by the eight- $(= 2^3)$-dimensional geometric algebra $\mathcal{G}_3$. To simplify expressions we introduce a new notation for the basis elements of grade 2 in *both* approaches to the geometric algebra $\mathcal{G}_3$.

$$\begin{aligned} l_1 &:= P_2 P_3 \\ l_2 &:= P_3 P_1 \\ l_3 &:= P_1 P_2 \end{aligned} \tag{53}$$

The homogeneous multivectors, together with their geometric interpretation, are shown in Table 8. According to Table 3, the translation of the basis elements of the geometric algebra $\mathcal{G}_3^+$ into the basis elements of the geometric algebra $\mathcal{G}_3^-$ is given by

$$\mathbb{1}^+ = \mathbb{I}^-, \tag{54a}$$

$$\{\eta_{11} P_1^+, \eta_{22} P_2^+, \eta_{33} P_3^+\} = \{l_1^-, l_2^-, l_3^-\}, \tag{54b}$$

$$\{\eta_{22}\eta_{33} l_1^+, \eta_{11}\eta_{33} l_2^+, \eta_{11}\eta_{22} l_3^+\} = \{-P_1^-, -P_2^-, -P_3^-\}, \tag{54c}$$

$$\eta_{11}\eta_{22}\eta_{33}\mathbb{I}^+ = -\mathbb{1}^-. \tag{54d}$$

A continuous one-to-one correspondence between the points and the lines of the planar field and the 1- and 2-vectors of the geometric algebra $\mathcal{G}_3$ or between the planes and the lines of the centric bundle and the 1- and 2-vectors of the geometric algebra $\mathcal{G}_3$ is a two dimensional projective coordinate system. Any choice of *four* generic fundamental elements defines the projective coordinate system (chapter VII of volume I in [VeYo]). In the 'plus' approach we start with

$\mathcal{G}_3^+$	**planar field**	**centric bundle**	$\mathcal{G}_3^-$
$\rho\mathbb{I}^+$			$\rho\mathbb{I}^-$
$\left.\begin{array}{r}\rho A_{\bar{1}}^+ = \mu_1^+ P_1^+ + \mu_2^+ P_2^+ + \\ + \mu_3^+ P_3^+\end{array}\right\}$	point	plane	$\left\{\begin{array}{l}\rho A_{\bar{2}}^- = \mu_1^- l_1^- + \mu_2^- l_2^- + \\ + \mu_3^- l_3^-\end{array}\right.$
$\left.\begin{array}{r}\rho A_{\bar{2}}^+ = \lambda_1^+ l_1^+ + \lambda_2^+ l_2^+ + \\ + \lambda_3^+ l_3^+\end{array}\right\}$	line	line	$\left\{\begin{array}{l}\rho A_{\bar{1}}^- = \lambda_1^- P_1^- + \lambda_2^- P_2^- + \\ + \lambda_3^- P_3^-\end{array}\right.$
$\rho\mathbb{1}^+$			$\rho\mathbb{1}^-$

Table 8: Geometric interpretation for the homogeneous multivectors of grade 1 and 2 in the planar field and the centric bundle. The grade of a geometric object depends on the chosen aspect to the geometric algebra $\mathcal{G}_3$. For example, in the planar field a point appears as a geometric object of grade 1 in $\mathcal{G}_3^+$, and as a geometric object of grade 2 in $\mathcal{G}_3^-$. The same holds for the plane in the centric bundle, and analogous for the lines in both geometries.

the plane of points or the bundle of planes, choose any four generic elements, and assign to them in any order the homogeneous coordinates $\rho(0,0,1)^+$, $\rho(0,1,0)^+$, $\rho(1,0,0)^+$, and $\rho(1,1,1)^+$. Any 1-vector $\rho A_{\bar{1}}^+ = \mu_1^+ P_1^+ + \mu_2^+ P_2^+ + \mu_3^+ P_3^+$ belongs to the point or plane with homogeneous coordinates $(\mu_1,\mu_2,\mu_3)^+$ or projective coordinates $(\frac{\mu_1}{\mu_3},\frac{\mu_2}{\mu_3})^+$, and any 2-vector $\rho A_{\bar{2}}^+ = \lambda_1^+ l_1^+ + \lambda_2^+ l_2^+ + \lambda_3^+ l_3^+$ belongs to the line with homogeneous line-coordinates $(\lambda_1,\lambda_2,\lambda_3)^+$ or projective line-coordinates $(\frac{\lambda_1}{\lambda_3},\frac{\lambda_2}{\lambda_3})^+$. In the dual approach to the same planar field or the same centric bundle we start with the plane of lines or bundle of lines respectively, choose any four generic lines, and assign to them in any order the homogeneous coordinates $\rho(0,0,1)^-$, $\rho(0,1,0)^-$, $\rho(1,0,0)^-$, and $\rho(1,1,1)^-$. Any 1-vector $\rho A_{\bar{1}}^- = \lambda_1^- P_1^- + \lambda_2^- P_2^- + \lambda_3^- P_3^-$ belongs to the line with homogeneous line-coordinates $(\lambda_1,\lambda_2,\lambda_3)^-$ or projective line-coordinates $(\frac{\lambda_1}{\lambda_3},\frac{\lambda_2}{\lambda_3})^-$, and any 2-vector $\rho A_{\bar{2}}^- = \mu_1^- l_1^- + \mu_2^- l_2^- + \mu_3^- l_3^-$ belongs to the point or plane with homogeneous coordinates $(\mu_1,\mu_2,\mu_3)^-$ or projective coordinates $(\frac{\mu_1}{\mu_3},\frac{\mu_2}{\mu_3})^-$.

The projective coordinates in the dual geometric algebras differ only by a sign depending on the signature chosen. From

$$\mu_1^+ P_1^+ + \mu_2^+ P_2^+ + \mu_3^+ P_3^+ = \mu_1^- l_1^- + \mu_2^- l_2^- + \mu_3^- l_3^- \tag{55}$$

we obtain the relations for the point- or plane-coordinates,

$$\begin{array}{lllllll} \mu_x^- & := & \frac{\mu_1^-}{\mu_3^-} & = & \left(\frac{\eta_{11}}{\eta_{33}}\right) \frac{\mu_1^+}{\mu_3^+} & =: & \left(\frac{\eta_{11}}{\eta_{33}}\right) \mu_x^+, \\ \mu_y^- & := & \frac{\mu_2^-}{\mu_3^-} & = & \left(\frac{\eta_{22}}{\eta_{33}}\right) \frac{\mu_2^+}{\mu_3^+} & =: & \left(\frac{\eta_{22}}{\eta_{33}}\right) \mu_y^+. \end{array} \tag{56}$$

And from

$$\lambda_1^+ l_1^+ + \lambda_2^+ l_2^+ + \lambda_3^+ l_3^+ = \lambda_1^- P_1^- + \lambda_2^- P_2^- + \lambda_3^- P_3^- \tag{57}$$

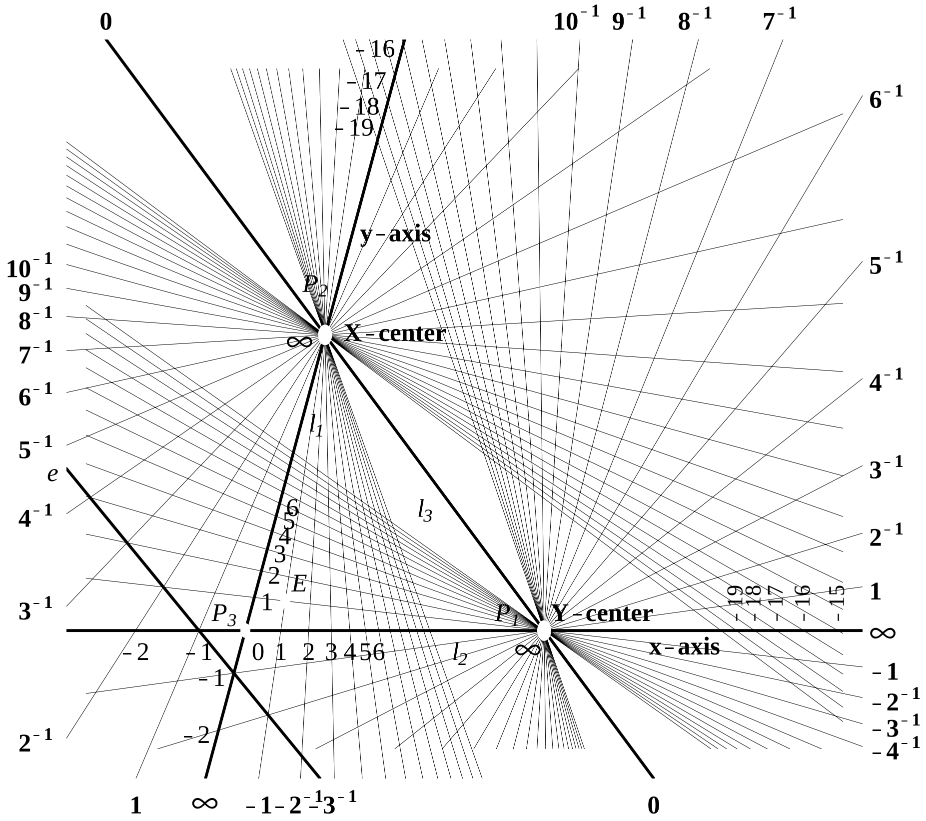

Figure 5: Projective coordinate system in a planar field. The projective point-coordinates are indicated along the x-axis (l_2) and the y-axis (l_1). The projective line-coordinates are indicated for the X-center (P_2) on the left side and the bottom, for the Y-center (P_1) on the right side and the top of the figure. The projective point-coordinates of E are $(1,1)^+$, and the projective line-coordinates of e are $(1,1)^+$. This coordinate system refers to the plus approach only.

$\mathcal{G}_4^+$	**space**	$\mathcal{G}_4^-$
$\rho\mathbb{1}^+$		$\rho\mathbb{I}^-$
$\rho A_1^+ = \mu_1^+ P_1^+ + \mu_2^+ P_2^+ + \mu_3^+ P_3^+ + \mu_4^+ P_4^+$	point	$\rho A_3^- = \mu_1^- \mathbf{P}_1^- + \mu_2^- \mathbf{P}_2^- + \mu_3^- \mathbf{P}_3^- + \mu_4^- \mathbf{P}_4^-$
$\rho A_2^+ = \lambda_1^+ l_1^+ + \lambda_2^+ l_2^+ + \lambda_3^+ l_3^+ + \lambda_4^+ l_4^+ + \lambda_5^+ l_5^+ + \lambda_6^+ l_6^+$	linear complex or a line	$\rho A_2^- = \lambda_1^- l_1^- + \lambda_2^- l_2^- + \lambda_3^- l_3^- + \lambda_4^- l_4^- + \lambda_5^- l_5^- + \lambda_6^- l_6^-$
$\rho A_3^+ = \sigma_1^+ \mathbf{P}_1^+ + \sigma_2^+ \mathbf{P}_2^+ + \sigma_3^+ \mathbf{P}_3^+ + \sigma_4^+ \mathbf{P}_4^+$	plane	$\rho A_1^- = \sigma_1^- P_1^- + \sigma_2^- P_2^- + \sigma_3^- P_3^- + \sigma_4^- P_4^-$
$\rho\mathbb{I}^+$		$\rho\mathbb{1}^-$

Table 9: Projective interpretation for homogeneous multivectors in space.

we get the relations for the line-coordinates,

$$\begin{aligned} \lambda_x^- &:= \frac{\lambda_1^-}{\lambda_3^-} = \left(\frac{\eta_{33}}{\eta_{11}}\right)\frac{\lambda_1^+}{\lambda_3^+} =: \left(\frac{\eta_{33}}{\eta_{11}}\right)\lambda_x^+, \\ \lambda_y^- &:= \frac{\lambda_2^-}{\lambda_3^-} = \left(\frac{\eta_{33}}{\eta_{22}}\right)\frac{\lambda_2^+}{\lambda_3^+} =: \left(\frac{\eta_{33}}{\eta_{22}}\right)\lambda_y^+. \end{aligned} \tag{58}$$

5.2.1 Planar Field

In a planar field, a generic multivector M of $\mathcal{G}_3$ decomposes into two different types of numbers, a point, and a line.

$$M^+ = \langle\text{scalar}\rangle_0 + \langle\text{point}\rangle_1 + \langle\text{line}\rangle_2 + \langle\text{pseudoscalar}\rangle_3 \tag{59a}$$

$$M^- = \langle\text{scalar}\rangle_0 + \langle\text{line}\rangle_1 + \langle\text{point}\rangle_2 + \langle\text{pseudoscalar}\rangle_3 \tag{59b}$$

Figure 5 shows an example for the coordinatization of a planar field.

5.2.2 Centric Bundle

In a centric bundle, a generic multivector M of $\mathcal{G}_3$ decomposes into two different types of numbers, a plane, and a line.

$$M^+ = \langle\text{scalar}\rangle_0 + \langle\text{plane}\rangle_1 + \langle\text{line}\rangle_2 + \langle\text{pseudoscalar}\rangle_3 \tag{60a}$$

$$M^- = \langle\text{scalar}\rangle_0 + \langle\text{line}\rangle_1 + \langle\text{plane}\rangle_2 + \langle\text{pseudoscalar}\rangle_3 \tag{60b}$$

Figure 6 shows an example for the coordinatization of a centric bundle.

5.3 Space

The space of points and the space of planes provide two illustrations for the three-dimensional projective space; the *space of linear complexes* is an example for a *five*-dimensional projective space containing the space of lines as a four-dimensional (quadratic) sub-manifold. All of them are represented algebraically by the 16-($= 2^4$)-dimensional geometric algebra $\mathcal{G}_4$. To simplify expressions we

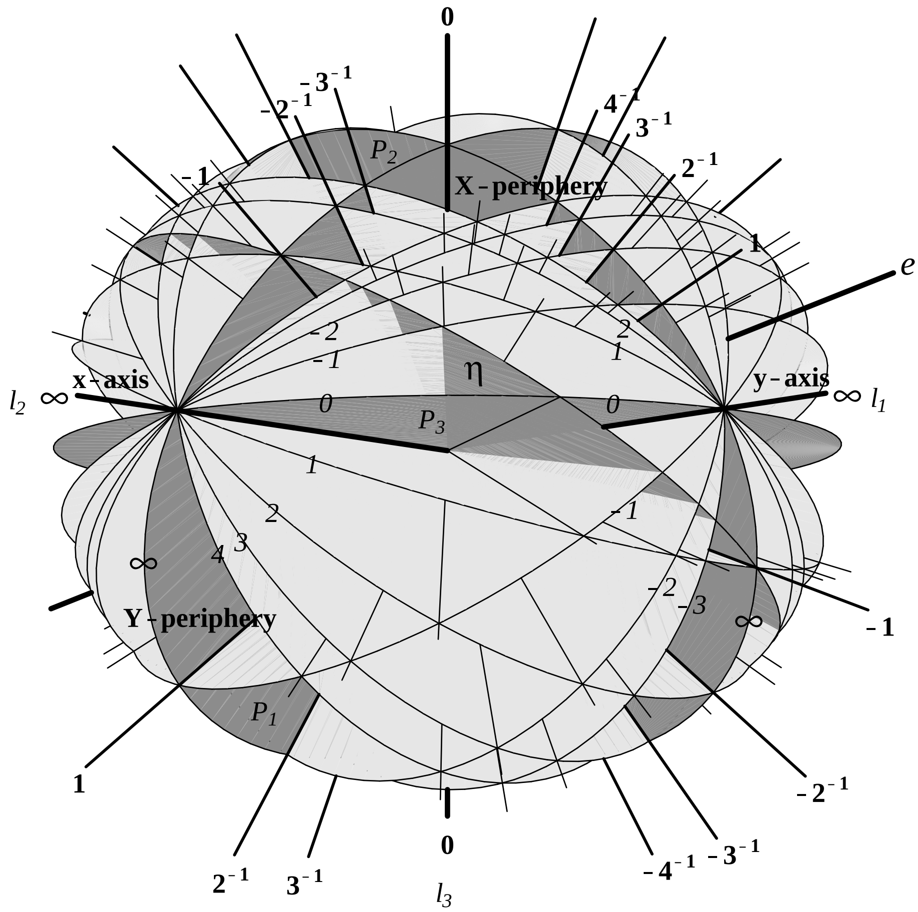

Figure 6: Projective coordinate system in a centric bundle. The projective plane-coordinates are indicated around the x-axis (l_2) and the y-axis (l_1). The projective line-coordinates are indicated along the circumferences of the X-periphery (P_2) and the Y-periphery (P_1). The projective plane-coordinates of η are $(1,1)^+$, and the projective line-coordinates of e are $(1,1)^+$. This coordinate systems only shows the plus approach.

introduce a new notation for the basis elements of grade 2 in both approaches to the geometric algebra $\mathcal{G}_4$,

$$\begin{aligned} l_1 &:= P_2P_3, & l_4 &:= P_4P_1, \\ l_2 &:= P_3P_1, & l_5 &:= P_4P_2, \\ l_3 &:= P_1P_2, & l_6 &:= P_4P_3, \end{aligned} \tag{61}$$

and also for the basis elements of grade 3.

$$\begin{aligned} \mathbf{P}_1 &:= P_2P_3P_4 \\ \mathbf{P}_2 &:= P_3P_4P_1 \\ \mathbf{P}_3 &:= P_4P_1P_2 \\ \mathbf{P}_4 &:= P_1P_2P_3 \end{aligned} \tag{62}$$

The homogeneous multivectors, together with their geometric interpretation, are shown in Table 9. According to Table 3, the translation of the basis elements of the geometric algebra $\mathcal{G}_4^+$ into the basis elements of the geometric algebra $\mathcal{G}_4^-$ is given by

$$\mathbb{1}^+ = \mathbb{I}^-, \tag{63a}$$

$$\begin{aligned} \{\eta_{11}P_1^+, -\eta_{22}P_2^+\} &= \{\mathbf{P}_1^-, \mathbf{P}_2^-\}, \\ \{\eta_{33}P_3^+, -\eta_{44}P_4^+\} &= \{\mathbf{P}_3^-, \mathbf{P}_4^-\}, \end{aligned} \tag{63b}$$

$$\begin{aligned} \{\eta_{22}\eta_{33}l_1^+, \eta_{11}\eta_{33}l_2^+\} &= \{l_4^-, l_5^-\}, \\ \{\eta_{11}\eta_{22}l_3^+, \eta_{11}\eta_{44}l_4^+\} &= \{l_6^-, l_1^-\}, \\ \{\eta_{22}\eta_{44}l_5^+, \eta_{33}\eta_{44}l_6^+\} &= \{l_2^-, l_3^-\}, \end{aligned} \tag{63c}$$

$$\begin{aligned} \{\eta_{22}\eta_{33}\eta_{44}\mathbf{P}_1^+, \eta_{11}\eta_{33}\eta_{44}\mathbf{P}_2^+\} &= \{P_1^-, -P_2^-\}, \\ \{\eta_{11}\eta_{22}\eta_{44}\mathbf{P}_3^+, \eta_{11}\eta_{22}\eta_{33}\mathbf{P}_4^+\} &= \{P_3^-, -P_4^-\}, \end{aligned} \tag{63d}$$

$$\eta_{11}\eta_{22}\eta_{33}\eta_{44}\mathbb{I}^+ = \mathbb{1}^-. \tag{63e}$$

A continuous one-to-one correspondence between the points, linear complexes, and planes and the 1-, 2-, and 3-vectors of the geometric algebra $\mathcal{G}_4$ provides a projective coordinate system in space. Any choice of either *five* generic fundamental points or *five* generic fundamental planes defines a projective coordinate system (chapter VII of volume I in [VeYo]). In the 'plus' approach we start with the space of points, choose any five generic points, and assign to them in any order the homogeneous coordinates $\rho(0,0,0,1)^+$, $\rho(0,0,1,0)^+$, $\rho(0,1,0,0)^+$, $\rho(1,0,0,0)^+$, and $\rho(1,1,1,1)^+$. Any 1-vector $\rho A_{\bar{1}}^+ = \mu_1^+P_1^+ + \mu_2^+P_2^+ + \mu_3^+P_3^+ + \mu_4^+P_4^+$ belongs to the point with homogeneous point-coordinates $(\mu_1, \mu_2, \mu_3, \mu_4)^+$ or projective point-coordinates $(\frac{\mu_1}{\mu_4}, \frac{\mu_2}{\mu_4}, \frac{\mu_3}{\mu_4})^+$, any 2-vector $\rho A_{\bar{2}}^+ = \lambda_1^+l_1^+ + \lambda_2^+l_2^+ + \lambda_3^+l_3^+ + \lambda_4^+l_4^+ + \lambda_5^+l_5^+ + \lambda_6^+l_6^+$ belongs to the linear complex with homogeneous complex-coordinates $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)^+$, and any 3-vector $\rho A_{\bar{3}}^+ = \sigma_1^+\mathbf{P}_1^+ + \sigma_2^+\mathbf{P}_2^+ + \sigma_3^+\mathbf{P}_3^+ + \sigma_4^+\mathbf{P}_4^+$ belongs to the plane with homogeneous plane-coordinates $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)^+$ or projective plane-coordinates $(\frac{\sigma_1}{\sigma_4}, \frac{\sigma_2}{\sigma_4}, \frac{\sigma_3}{\sigma_4})^+$; see Figure 7. In the 'minus' approach, we start with the space

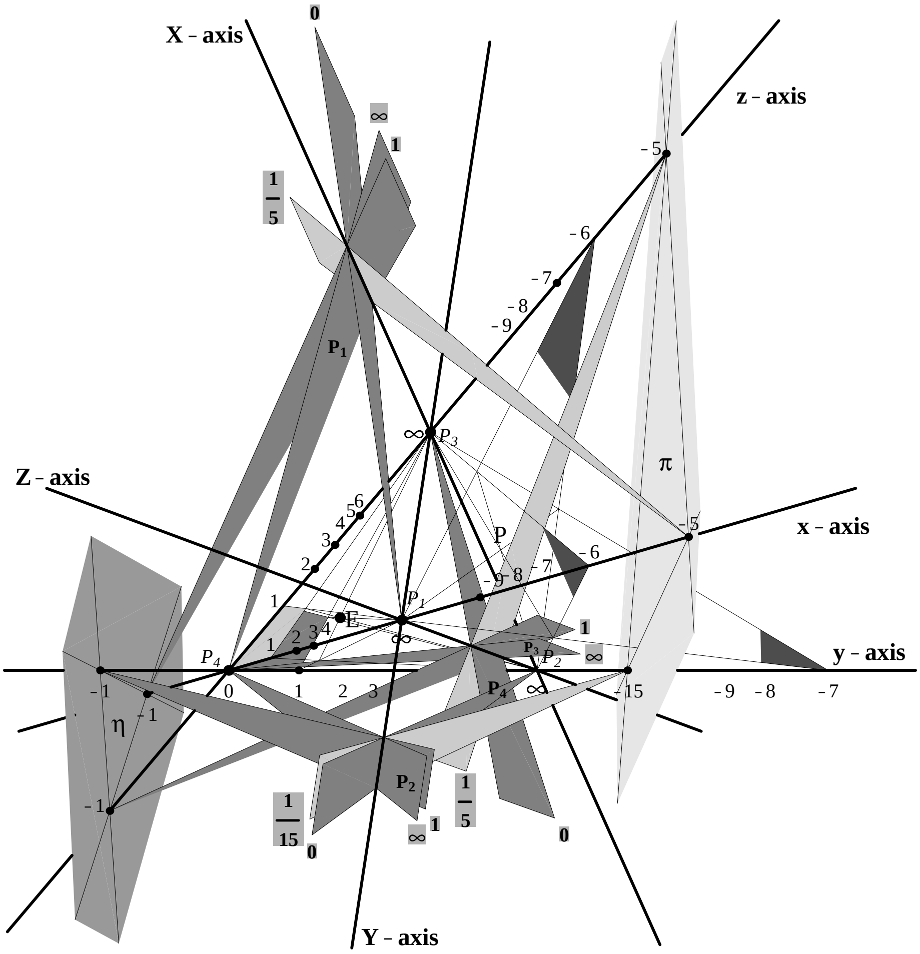

Figure 7: Projective coordinate system in space. The projective point-coordinates are indicated along the x-axis ($l_4 = P_4P_1$), the y-axis ($l_5 = P_4P_2$), and the z-axis ($l_6 = P_4P_3$). $\mathbf{P}_4$ represents the plane at infinity in the space of points. The projective plane-coordinates are indicated around the X-axis ($l_1 = P_2P_3$), the Y-axis ($l_2 = P_3P_1$), and the Z-axis ($l_3 = P_1P_2$), and are put on a gray background. P_4 represents the point at infinity in the space of planes. The projective point-coordinates of E and P are $(1, 1, 1)^+$ and $(-6, -7, -6)^+$ respectively. The projective plane-coordinates of η and π are $(1, 1, 1)^+$ and $(\frac{1}{5}, \frac{1}{15}, \frac{1}{5})^+$ respectively. This figure refers to the plus approach only.

of planes, choose any five generic planes, and assign to them in any order the homogeneous coordinates $\rho(0,0,0,1)^-$, $\rho(0,0,1,0)^-$, $\rho(0,1,0,0)^-$, $\rho(1,0,0,0)^-$, and $\rho(1,1,1,1)^-$. Any 1-vector $\rho A_{\bar{1}}^- = \mu_1^- P_1^- + \mu_2^- P_2^- + \mu_3^- P_3^- + \mu_4^- P_4^-$ belongs to the plane with homogeneous plane-coordinates $(\mu_1, \mu_2, \mu_3, \mu_4)^-$ or projective plane-coordinates $(\frac{\mu_1}{\mu_4}, \frac{\mu_2}{\mu_4}, \frac{\mu_3}{\mu_4})^-$, any 2-vector $\rho A_{\bar{2}}^- = \lambda_1^- l_1^- + \lambda_2^- l_2^- + \lambda_3^- l_3^- + \lambda_4^- l_4^- + \lambda_5^- l_5^- + \lambda_6^- l_6^-$ belongs to the linear complex with homogeneous complex-coordinates $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)^-$, and any 3-vector $\rho A_{\bar{3}}^- = \sigma_1^- \mathbf{P}_1^- + \sigma_2^- \mathbf{P}_2^- + \sigma_3^- \mathbf{P}_3^- + \sigma_4^- \mathbf{P}_4^-$ belongs to the point with homogeneous point-coordinates $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)^-$ or projective point-coordinates $(\frac{\sigma_1}{\sigma_4}, \frac{\sigma_2}{\sigma_4}, \frac{\sigma_3}{\sigma_4})^-$.

The projective coordinates of points and planes in the dual geometric algebras differ only by a sign which depends on the chosen signature. From

$$\mu_1^+ P_1^+ + \mu_2^+ P_2^+ + \mu_3^+ P_3^+ + \mu_4^+ P_4^+ = \mu_1^- \mathbf{P}_1^- + \mu_2^- \mathbf{P}_2^- + \mu_3^- \mathbf{P}_3^- + \mu_4^- \mathbf{P}_4^- \quad (64)$$

we get the relations for the point-coordinates,

$$\begin{aligned}
\mu_x^- &:= \frac{\mu_1^-}{\mu_4^-} = -\left(\frac{\eta_{11}}{\eta_{44}}\right)\frac{\mu_1^+}{\mu_4^+} =: -\left(\frac{\eta_{11}}{\eta_{44}}\right)\mu_x^+, \\
\mu_y^- &:= \frac{\mu_2^-}{\mu_4^-} = \left(\frac{\eta_{22}}{\eta_{44}}\right)\frac{\mu_2^+}{\mu_4^+} =: \left(\frac{\eta_{22}}{\eta_{44}}\right)\mu_y^+, \\
\mu_z^- &:= \frac{\mu_3^-}{\mu_4^-} = -\left(\frac{\eta_{33}}{\eta_{44}}\right)\frac{\mu_3^+}{\mu_4^+} =: -\left(\frac{\eta_{33}}{\eta_{44}}\right)\mu_z^+.
\end{aligned} \quad (65)$$

And from

$$\sigma_1^+ \mathbf{P}_1^+ + \sigma_2^+ \mathbf{P}_2^+ + \sigma_3^+ \mathbf{P}_3^+ + \sigma_4^+ \mathbf{P}_4^+ = \sigma_1^- P_1^- + \sigma_2^- P_2^- + \sigma_3^- P_3^- + \sigma_4^- P_4^- \quad (66)$$

we obtain the relations for the plane-coordinates,

$$\begin{aligned}
\sigma_x^- &:= \frac{\sigma_1^-}{\sigma_4^-} = -\left(\frac{\eta_{11}}{\eta_{44}}\right)\frac{\sigma_1^+}{\sigma_4^+} =: -\left(\frac{\eta_{11}}{\eta_{44}}\right)\sigma_x^+, \\
\sigma_y^- &:= \frac{\sigma_2^-}{\sigma_4^-} = \left(\frac{\eta_{22}}{\eta_{44}}\right)\frac{\sigma_2^+}{\sigma_4^+} =: \left(\frac{\eta_{22}}{\eta_{44}}\right)\sigma_y^+, \\
\sigma_z^- &:= \frac{\sigma_3^-}{\sigma_4^-} = -\left(\frac{\eta_{33}}{\eta_{44}}\right)\frac{\sigma_3^+}{\sigma_4^+} =: -\left(\frac{\eta_{33}}{\eta_{44}}\right)\sigma_z^+.
\end{aligned} \quad (67)$$

For the geometric interpretation of the line-coordinates and the coordinates of linear complexes we refer to the book of H.-J. Stoß [St].

In space, a generic multivector M of $\mathcal{G}_4$ decomposes into two different types of numbers, a point, a (special) linear complex, and a plane.

$$M^+ = \langle\text{scalar}\rangle_0 + \langle\text{point}\rangle_1 + \langle\text{complex}\rangle_2 + \langle\text{plane}\rangle_3 + \langle\text{p.-scalar}\rangle_4 \quad (68a)$$
$$M^- = \langle\text{scalar}\rangle_0 + \langle\text{plane}\rangle_1 + \langle\text{complex}\rangle_2 + \langle\text{point}\rangle_3 + \langle\text{p.-scalar}\rangle_4 \quad (68b)$$

6 Basic Relations and Operations

In the first part of this section we review the incidence relations of homogeneous multivectors in space, introduce the linear complex and apply the incidence relation also to the non-homogeneous multivector of a point-plane pair. The second part reviews the operations of connection and intersection.

6.1 Incidence

The incidence relations are formulated as identities in geometric algebra.

Definition 1 *Two multivectors A and B are* incident *if and only if $A \wedge B = 0$ and $A \vee B = 0$.*

$$A_{\bar{1}}^{+} \wedge B_{\bar{1}}^{+} = 0, \quad A_{\bar{1}}^{+} \vee B_{\bar{1}}^{+} = 0 \quad \Leftrightarrow \quad A_{\bar{1}}^{+} = \beta B_{\bar{1}}^{+} \tag{69a}$$

$$A_{\bar{1}}^{-} \vee B_{\bar{1}}^{-} = 0, \quad A_{\bar{1}}^{-} \wedge B_{\bar{1}}^{-} = 0 \quad \Leftrightarrow \quad A_{\bar{1}}^{-} = \beta B_{\bar{1}}^{-} \tag{69b}$$

The incidence relation between points (69a) and planes (69b), respectively, is reflexive. Thus, a point lies in itself and a plane passes through itself.

Theorem 1 *The following statements are equivalent: i) $A_{\bar{2}}$ represents a line; ii) $A_{\bar{2}} \wedge A_{\bar{2}} = 0$; iii) $A_{\bar{2}} \vee A_{\bar{2}} = 0$; and iv) $A_{\bar{2}}$ is incident with itself.*

Proof.

i) $\Rightarrow$ ii) Any line may be represented by the connection of two distinct points $B_{\bar{1}}^{+}$ and $C_{\bar{1}}^{+}$: $A_{\bar{2}}^{+} = B_{\bar{1}}^{+} \wedge C_{\bar{1}}^{+}$. It follows that $A_{\bar{2}}^{+} \wedge A_{\bar{2}}^{+} = 0$. In the dual approach any line may be represented by the intersection of two distinct planes $B_{\bar{1}}^{-}$ and $C_{\bar{1}}^{-}$: $A_{\bar{2}}^{-} = B_{\bar{1}}^{-} \vee C_{\bar{1}}^{-}$. It follows that $A_{\bar{2}}^{-} \wedge A_{\bar{2}}^{-} = 0$.

ii) $\Rightarrow$ iii) $A_{\bar{2}} \vee A_{\bar{2}} = (-1)^{q}(A_{\bar{2}} \wedge A_{\bar{2}})^{\sim} \qquad \forall A_{\bar{2}} \in \mathcal{G}_{p,q}^{2}$.

iii) $\Rightarrow$ iv) Definition 1.

iv) $\Rightarrow$ i): The solutions of the incidence equations $A_{\bar{2}} \wedge X_{\bar{1}} = 0$ and $A_{\bar{2}} \vee X_{\bar{1}} = 0$ have the form of a pencil of points or of a pencil of planes: $X_{\bar{1}} = \mu_1 B_{\bar{1}} + \mu_2 C_{\bar{1}}$. Thus, $A_{\bar{2}}$ represents a line.

□

The incidence relation between 2-vectors is *not* reflexive in general. A direct geometric interpretation for 2-vectors $A_{\bar{2}}$ with $A_{\bar{2}} \wedge A_{\bar{2}} \neq 0$ and $A_{\bar{2}} \vee A_{\bar{2}} \neq 0$ does not exist. Nevertheless, any 2-vector is incident with certain lines in space and is thus characterized by them.

Definition 2 *The locus of all lines that are incident with a given 2-vector $A_{\bar{2}}$ is called* a linear complex of $A_{\bar{2}}$ *if $A_{\bar{2}} \wedge A_{\bar{2}} \neq 0$ and $A_{\bar{2}} \vee A_{\bar{2}} \neq 0$, and* a special linear complex of $A_{\bar{2}}$ *if $A_{\bar{2}} \wedge A_{\bar{2}} = 0$ and $A_{\bar{2}} \vee A_{\bar{2}} = 0$.*

Instead of the correct expression '(special) linear complex of $A_{\bar{2}}$' we will use the shorter but less precise expression '(special) linear complex $A_{\bar{2}}$'.

Theorem 2 *A (special) linear complex is a three-dimensional subset in the space of lines.*

Proof. A line $B_{\bar{2}}$ is incident with a (special) linear complex $A_{\bar{2}}$ if and only if $A_{\bar{2}} \wedge B_{\bar{2}} = 0$ and $A_{\bar{2}} \vee B_{\bar{2}} = 0$. Thus, three degrees of freedom are left to the line $B_{\bar{2}}$. □

Theorem 3 *a) Given a linear complex $A_{\bar{2}}$. Every bundle in space contains one and only one pencil of lines belonging to the linear complex $A_{\bar{2}}$, and every field in space contains one and only one pencil of lines belonging to the linear complex $A_{\bar{2}}$.*
b) Given a special linear complex $A_{\bar{2}}$. The same statement as in a) holds for all bundles and fields whose points and planes, respectively, are not incident with the line $A_{\bar{2}}$. In case they are incident, all the lines of the bundle and all the lines of the field belong to the special linear complex $A_{\bar{2}}$.

Proof. a) Given any point $B^+_{\bar{1}}$ and the linear complex $A_{\bar{2}}$. The solutions of the system $A^+_{\bar{2}} \wedge X^+_{\bar{2}} = 0$, $A^+_{\bar{2}} \vee X^+_{\bar{2}} = 0$, $B^+_{\bar{1}} \wedge X^+_{\bar{2}} = 0$, $B^+_{\bar{1}} \vee X^+_{\bar{2}} = 0$ form a pencil of lines with center $B^+_{\bar{1}}$. Given any plane $B^-_{\bar{1}}$ and the linear complex $A_{\bar{2}}$. The solutions of the system $A^-_{\bar{2}} \vee X^-_{\bar{2}} = 0$, $A^-_{\bar{2}} \wedge X^-_{\bar{2}} = 0$, $B^-_{\bar{1}} \vee X^-_{\bar{2}} = 0$, $B^-_{\bar{1}} \wedge X^-_{\bar{2}} = 0$ form a pencil of lines in the plane $B^-_{\bar{1}}$.
b) In case the point $B^+_{\bar{1}}$ or the plane $B^-_{\bar{1}}$ belongs to the line $A_{\bar{2}}$, the incidence equations $A_{\bar{2}} \wedge X_{\bar{2}} = 0$ and $A_{\bar{2}} \vee X_{\bar{2}} = 0$ hold trivially for all $X_{\bar{2}}$ passing through $B^+_{\bar{1}}$ or lying in $B^-_{\bar{1}}$, respectively. Otherwise see case a). □

Null-polarities ([Co], p. 69f) are intimately connected with non-special linear complexes.

Theorem 4 *Every null-polarity determines one and only one non-special linear complex, and every non-special linear complex determines one and only one null-polarity. The lines of the non-special linear complex are the only invariant elements of the corresponding null-polarity.*

Proof. See chapter XI of volume I in [VeYo]. □

Definition 3 *Two non-special linear complexes $A_{\bar{2}}$ and $B_{\bar{2}}$ are* null-invariant to each other *if and only if the linear complex $A_{\bar{2}}$ is invariant under the null-polarity of $B_{\bar{2}}$ or, which is the same, if and only if the linear complex $B_{\bar{2}}$ is invariant under the null-polarity of $A_{\bar{2}}$. In case at least one of the linear complexes is special, say $A_{\bar{2}}$, the linear complexes $A_{\bar{2}}$ and $B_{\bar{2}}$ are null-invariant, if and only if the line $A_{\bar{2}}$ belongs to the (special) linear complex $B_{\bar{2}}$.*

$$A_{\bar{2}} \wedge B_{\bar{2}} = 0, \qquad A_{\bar{2}} \vee B_{\bar{2}} = 0 \tag{69c}$$

The equations (69c) express the incidence relations for lines and linear complexes. In detail we have:

1. If $A_{\bar{2}} = \beta B_{\bar{2}}$, then $A_{\bar{2}}$ represents a line.
2. If $A_{\bar{2}} \neq \beta B_{\bar{2}}$, then we distinguish the following cases:
   (a) If $A_{\bar{2}} \wedge A_{\bar{2}} = 0$, $B_{\bar{2}} \wedge B_{\bar{2}} = 0$, and equations (69c) are satisfied, then the lines $A_{\bar{2}}$ and $B_{\bar{2}}$ meet, i.e., they pass through a common point and lie in a common plane.

(b) $A_{\bar{2}} \wedge A_{\bar{2}} \neq 0$ and $B_{\bar{2}} \wedge B_{\bar{2}} = 0$.
The line $B_{\bar{2}}$ belongs to the linear complex $A_{\bar{2}}$.

(c) $A_{\bar{2}} \wedge A_{\bar{2}} = 0$ and $B_{\bar{2}} \wedge B_{\bar{2}} \neq 0$.
The line $A_{\bar{2}}$ belongs to the linear complex $B_{\bar{2}}$.

(d) $A_{\bar{2}} \wedge A_{\bar{2}} \neq 0$ and $B_{\bar{2}} \wedge B_{\bar{2}} \neq 0$.
The linear complexes $A_{\bar{2}}$ and $B_{\bar{2}}$ are null-invariant.

$$A_{\bar{2}}^{+} \wedge B_{\bar{1}}^{+} = 0, \qquad A_{\bar{2}}^{+} \vee B_{\bar{1}}^{+} = 0 \tag{69d}$$

The second equation of (69d) is trivially satisfied.

1. If the 2-vector $A_{\bar{2}}^{+}$ represents a line, the equations (69d) require the point $B_{\bar{1}}^{+}$ and the line $A_{\bar{2}}^{+}$ to be incident. Thus, the set of all points lying in a fixed line forms an one-dimensional *pencil of points.* The set of all lines passing through a fixed point forms a two-dimensional *bundle of lines.*

2. Direct computation shows that the equations (69d) have no solutions other than 0 in case $A_{\bar{2}}^{+}$ represents a linear complex. Thus, there are no points lying in a linear complex!

$$A_{\bar{2}}^{-} \vee B_{\bar{1}}^{-} = 0, \qquad A_{\bar{2}}^{-} \wedge B_{\bar{1}}^{-} = 0 \tag{69e}$$

The second equation of (69e) is trivially true.

1. If the 2-vector $A_{\bar{2}}^{-}$ represents a line, the equations (69e) require the plane $B_{\bar{1}}^{-}$ and the line $A_{\bar{2}}^{-}$ to be incident. Thus, the set of all planes passing through a fixed line forms an one-dimensional *pencil of planes.* The set of all lines lying in a fixed plane form a two-dimensional *field of lines.*

2. Direct computation shows that the equations (69e) do not have solutions other than 0 in case $A_{\bar{2}}^{-}$ represents a linear complex. Thus, there are no planes passing through a linear complex!

$$A_{\bar{3}} \wedge B_{\bar{1}} = 0, \qquad A_{\bar{3}} \vee B_{\bar{1}} = 0 \tag{69f}$$

The equations (69f) require the plane $A_{\bar{3}}^{+}$, $B_{\bar{1}}^{-}$ and the point $B_{\bar{1}}^{+}$, $A_{\bar{3}}^{-}$, respectively, to be incident. The set of all points lying in a fixed plane forms a two-dimensional *plane of points*, and the set of all planes passing through a fixed point forms a two-dimensional *bundle of planes.*

Let us now consider incident multivectors. What is the condition for two generic point-plane pairs $A = \langle A\rangle_1 + \langle A\rangle_3$ and $B = \langle B\rangle_1 + \langle B\rangle_3$ to be incident?

$$\left.\begin{array}{l} A \wedge B = 0 \\ A \vee B = 0 \end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{l} \langle A\rangle_1 = \alpha\langle B\rangle_1 \\ \langle A\rangle_3 = \beta\langle B\rangle_3 \\ \langle A\rangle_1 \wedge \langle A\rangle_3 = \langle B\rangle_1 \vee \langle B\rangle_3 = 0 \end{array}\right. \tag{69g}$$

They are incident if and only if A and B represent the same incident point-plane pair! Thus, the incidence relation between incident point-plane pairs is reflexive as for points (69a) and planes (69b).

An incident point-plane pair $A = \langle A\rangle_1 + \langle A\rangle_3$ and a line $B_{\bar{2}}$ are incident,

$$(\langle A\rangle_1 + \langle A\rangle_3) \wedge B_{\bar{2}} = 0, \qquad (\langle A\rangle_1 + \langle A\rangle_3) \vee B_{\bar{2}} = 0, \tag{69h}$$

if and only if the line $B_{\bar{2}}$ passes through the point $\langle A\rangle_{\bar{1}}^{+}$ ($\langle A\rangle_{\bar{3}}^{-}$) and lies in the plane $\langle A\rangle_{\bar{3}}^{+}$ ($\langle A\rangle_{\bar{1}}^{-}$) simultaneously. The set of all lines that are incident with a fixed incident point-plane pair forms an one-dimensional *pencil of lines.*

The equations (69a), (69b), (69c), and (69g) express the incidence relations for pure elements. The equations (69d), (69e), (69f), and (69h) show the incidence relations between different pure elements according to the diagram of Figure 1.

6.2 Connection and Intersection

Definition 4 *The* connection S *of two multivectors* A *and* B *is* $S = A \wedge B$ *if* A *and* B *are not incident. The* intersection S *of two multivectors* A *and* B *is* $S = A \vee B$ *if* A *and* B *are not incident.*

The equations (70) review the relations of connection and intersection in space. We limit our attention to homogeneous multivectors.

$$S_{\bar{2}}^{+} = A_{\bar{1}}^{+} \wedge B_{\bar{1}}^{+} \neq 0, \quad S_{\bar{2}}^{-} = A_{\bar{1}}^{-} \vee B_{\bar{1}}^{-} \neq 0 \tag{70a}$$

The connection of two distinct points $A_{\bar{1}}^{+}$ and $B_{\bar{1}}^{+}$ is one and only one line $S_{\bar{2}}^{+}$ ($S_{\bar{2}}^{+} \wedge S_{\bar{2}}^{+} = 0$). The intersection of two distinct planes $A_{\bar{1}}^{-}$ and $B_{\bar{1}}^{-}$ is one and only one line $S_{\bar{2}}^{-}$ ($S_{\bar{2}}^{-} \vee S_{\bar{2}}^{-} = 0$).

$$S_{\bar{4}}^{+} = A_{\bar{2}}^{+} \wedge B_{\bar{2}}^{+} \neq 0, \quad S_{\bar{4}}^{-} = A_{\bar{2}}^{-} \vee B_{\bar{2}}^{-} \neq 0 \tag{70b}$$

The connection of two not necessarily distinct (special) linear complexes $A_{\bar{2}}^{+}$ and $B_{\bar{2}}^{+}$, that are not null-invariant to each other, is one and only one pseudoscalar $S_{\bar{4}}^{+}$. The intersection of two not necessarily distinct (special) linear complexes $A_{\bar{2}}^{-}$ and $B_{\bar{2}}^{-}$, that are not null-invariant to each other, is one and only one pseudoscalar $S_{\bar{4}}^{-}$.

$$A_{\bar{3}}^{+} \wedge B_{\bar{3}}^{+} = 0, \quad A_{\bar{3}}^{-} \vee B_{\bar{3}}^{-} = 0, \quad \forall A_{\bar{3}}, B_{\bar{3}} \in \mathcal{G}_{p,q}^{3} \tag{70c}$$

There is no connection between any two planes $A_{\bar{3}}^{+}$ and $B_{\bar{3}}^{+}$, and no intersection between any two points $A_{\bar{3}}^{-}$ and $B_{\bar{3}}^{-}$.

$$S_{\bar{3}}^{+} = A_{\bar{2}}^{+} \wedge B_{\bar{1}}^{+} \neq 0, \quad S_{\bar{3}}^{-} = A_{\bar{2}}^{-} \vee B_{\bar{1}}^{-} \neq 0 \tag{70d}$$

plane of points		**plane of lines**	$\mathcal{G}_3^+$		$\mathcal{G}_3^-$
			$A^+_{\bar{0}}$	$\leftrightarrow$	$A^-_{\bar{0}}$
point	$\leftrightarrow$	line	$A^+_{\bar{1}}$	$\leftrightarrow$	$A^-_{\bar{1}}$
line	$\leftrightarrow$	point	$A^+_{\bar{2}}$	$\leftrightarrow$	$A^-_{\bar{2}}$
			$A^+_{\bar{3}}$	$\leftrightarrow$	$A^-_{\bar{3}}$
lying in	$\leftrightarrow$	passing through	$\left.\begin{array}{r} A^+ \wedge B^+ = 0 \\ \text{and} \\ A^+ \vee B^+ = 0 \end{array}\right\}$	$\leftrightarrow$	$\left\{\begin{array}{l} A^- \vee B^- = 0 \\ \text{and} \\ A^- \wedge B^- = 0 \end{array}\right.$
passing through	$\leftrightarrow$	lying in	$\left.\begin{array}{r} A^+ \wedge B^+ = 0 \\ \text{and} \\ A^+ \vee B^+ = 0 \end{array}\right\}$	$\leftrightarrow$	$\left\{\begin{array}{l} A^- \vee B^- = 0 \\ \text{and} \\ A^- \wedge B^- = 0 \end{array}\right.$
connect	$\leftrightarrow$	intersect	$\wedge$	$\leftrightarrow$	$\vee$
intersect	$\leftrightarrow$	connect	$\vee$	$\leftrightarrow$	$\wedge$
			$\cdot$	$\leftrightarrow$	$\circ$
			$\circ$	$\leftrightarrow$	$\cdot$
			(juxtaposition)	$\leftrightarrow$	$*$
			$*$	$\leftrightarrow$	(juxtaposition)

Table 10: The principle of duality in the field $\mathcal{G}_3$.

The connection of a (special) linear complex $A^+_{\bar{2}}$ and a point $B^+_{\bar{1}}$ is one and only one plane $S^+_{\bar{3}}$. The intersection of a (special) linear complex $A^-_{\bar{2}}$ and a plane $B^-_{\bar{1}}$ is one and only one point $S^-_{\bar{3}}$.

$$S^+_{\bar{4}} = A^+_{\bar{3}} \wedge B^+_{\bar{1}} \neq 0, \quad S^-_{\bar{4}} = A^-_{\bar{3}} \vee B^-_{\bar{1}} \neq 0 \tag{70e}$$

The connection of a plane $A^+_{\bar{3}}$ and a point $B^+_{\bar{1}}$ is one and only one pseudoscalar $S^+_{\bar{4}}$. The intersection of a point $A^-_{\bar{3}}$ and a plane $B^-_{\bar{1}}$ is one and only one pseudoscalar $S^-_{\bar{4}}$.

$$A^+_{\bar{3}} \wedge B^+_{\bar{2}} = 0, \quad A^-_{\bar{3}} \vee B^-_{\bar{2}} = 0 \quad \forall A_{\bar{3}} \in \mathcal{G}^3_{p,q}, \quad \forall B_{\bar{2}} \in \mathcal{G}^2_{p,q} \tag{70f}$$

There is no connection defined between a plane $A^+_{\bar{3}}$ and a (special) linear complex $B^+_{\bar{2}}$. Equivalently there is no intersection between a point $A^-_{\bar{3}}$ and a (special) linear complex $B^-_{\bar{2}}$.

7 Principle of Duality

The principle of duality was first stated by the French mathematicians J. V. Poncelet and J.-D. Gergonne (1826) in the first quarter of the nineteenth century. In Germany, A. F. Möbius rediscovered the principle of duality independently and used the homogeneous coordinates in his book *Das barycentrische Calcul* (1827) for the first time. J. Plücker discovered the line- and plane-coordinates

bundle of planes		bundle of lines	$\mathcal{G}_3^+$		$\mathcal{G}_3^-$
			A_0^+	$\leftrightarrow$	A_0^-
plane	$\leftrightarrow$	line	A_1^+	$\leftrightarrow$	A_1^-
line	$\leftrightarrow$	plane	A_2^+	$\leftrightarrow$	A_2^-
			A_3^+	$\leftrightarrow$	A_3^-
lying in	$\leftrightarrow$	passing through	$\left.\begin{array}{r} A^+ \wedge B^+ = 0 \\ \text{and} \\ A^+ \vee B^+ = 0 \end{array}\right\}$	$\leftrightarrow$	$\left\{\begin{array}{l} A^- \vee B^- = 0 \\ \text{and} \\ A^- \wedge B^- = 0 \end{array}\right.$
passing through	$\leftrightarrow$	lying in	$\left.\begin{array}{r} A^+ \wedge B^+ = 0 \\ \text{and} \\ A^+ \vee B^+ = 0 \end{array}\right\}$	$\leftrightarrow$	$\left\{\begin{array}{l} A^- \vee B^- = 0 \\ \text{and} \\ A^- \wedge B^- = 0 \end{array}\right.$
connect	$\leftrightarrow$	intersect	$\wedge$	$\leftrightarrow$	$\vee$
intersect	$\leftrightarrow$	connect	$\vee$	$\leftrightarrow$	$\wedge$
			$\cdot$	$\leftrightarrow$	$\circ$
			$\circ$	$\leftrightarrow$	$\cdot$
			(juxtaposition)	$\leftrightarrow$	$*$
			$*$	$\leftrightarrow$	(juxtaposition)

Table 11: The principle of duality in the bundle $\mathcal{G}_3$.

that gave a firm analytic base to the principle of duality. F. Klein removed the last vestiges of Euclidean geometry by showing every metric geometry—non-Euclidean or Euclidean—with constant curvature to be embedded in projective geometry ([Kl]). He also provided the algebraic foundation for projective geometry. F. Klein considered the principle of duality as a significant step in the development of mathematics (and natural sciences)."Die Auffindung des Dualitätsprinzips, das von unserem heutigen Standpunkt aus nicht allzu tiefliegend erscheint, stellte eine wesentliche wissenschaftliche Leistung dar. Man erkennt dies am besten daran, dass rund 150 Jahre nach der Auffindung des Pascalschen Satzes vergangen sind, ehe der Satz des Brianchon gefunden wurde, der sich doch mit Hilfe des Dualitätsprinzips durch eine unmittelbare Übertragung aus dem ersten Satz ableiten lässt." ([Kl], p. 38)

To justify the principle of duality in synthetic projective geometry, the *method of formal inference* is used. Starting from a system of explicitly stated axioms that imply their own duals, half of the theorems appear self-evident. If one theorem is proven by tracing it back to some axioms, the dual theorem is immediately true by formal inference, i.e., it traces back to the dual axioms belonging to the system of explicitly stated axioms. This is one of the important advantages of this method. The books of O. Veblen and J. W. Young [VeYo], L. Locher [Lo40, Lo57], and H. S. M. Coxeter [Co] all approach projective geometry from the vantage point of formal inference.

"The principle of duality in the *plane* affirms that every definition remains significant, and every theorem remains true, when we interchange 'point' and

space of points		space of planes	$\mathcal{G}_4^+$		$\mathcal{G}_4^-$
			A_0^+	↔	A_0^-
point	↔	plane	A_1^+	↔	A_1^-
line	↔	line	A_2^+	↔	A_2^-
plane	↔	point	A_3^+	↔	A_3^-
			A_4^+	↔	A_4^-
lying in	↔	passing through	$A^+ \wedge B^+ = 0$ and $A^+ \vee B^+ = 0$	↔	$A^- \vee B^- = 0$ and $A^- \wedge B^- = 0$
passing through	↔	lying in	$A^+ \wedge B^+ = 0$ and $A^+ \vee B^+ = 0$	↔	$A^- \vee B^- = 0$ and $A^- \wedge B^- = 0$
connect	↔	intersect	$\wedge$	↔	$\vee$
intersect	↔	connect	$\vee$	↔	$\wedge$
			$\cdot$	↔	$\circ$
			$\circ$	↔	$\cdot$
			(juxtaposition)	↔	$*$
			$*$	↔	(juxtaposition)

Table 12: The principle of duality in space $\mathcal{G}_4$.

'line', and make a few consequent alterations in wording." ([Co], p. 26) The 'consequent alterations in wording' are shown on the left side in Table 10. To illustrate the principle of duality in the field, we use the famous *theorem of Desargues* (Theorem 5) which is self-dual (Theorem 6).

Let $\Delta = \{A, B, C\}$ and $\Delta' = \{A', B', C'\}$ be two generic triangles with the sides $\delta = \{a, b, c\}$ and $\delta' = \{a', b', c'\}$.

Theorem 5 *If and only if Δ and Δ' are perspective from a point Z (i.e., the three connecting lines $p := AA'$, $q := BB'$, and $r := CC'$ pass through the common point Z), the three intersecting points $P := aa'$, $Q := bb'$, and $R := cc'$ are perspective from a line z (i.e., the points P, Q, and R lie on the common line z).*

Theorem 6 *If and only if δ and δ' are perspective from a line z (i.e., the three intersecting points $P := aa'$, $Q := bb'$, and $R := cc'$ lie on the common line z), the three lines $p := AA'$, $q := BB'$, and $r := CC'$ are perspective from a point Z (i.e., the lines p, q, and r pass through the common point Z.)*

The principle of duality in the *bundle* allows for the analogous interchange of 'plane' and 'line' and a few consequent alterations shown on the left side in Table 11. Theorem 7 and 8 provide an example for the duality in the bundle.

Let $\Delta = \{A, B, C\}$ and $\Delta' = \{A', B', C'\}$ be two generic 'triplanes' with the edges $\delta = \{a, b, c\}$ and $\delta' = \{a', b', c'\}$.

Theorem 7 *If and only if* Δ *and* Δ' *are perspective from a plane* Z *(i.e., the three intersecting lines* $p := AA'$*,* $q := BB'$*, and* $r := CC'$ *lie in the common plane* Z*), the three connecting planes* $P := aa'$*,* $Q := bb'$*, and* $R := cc'$ *are perspective from a line* z *(i.e., the planes* P*,* Q*, and* R *pass through the common line* z*).*

Theorem 8 *If and only if* δ *and* δ' *are perspective from a line* z *(i.e., the three connecting planes* $P := aa'$*,* $Q := bb'$*, and* $R := cc'$ *pass through the common line* z*), the three lines* $p := AA'$*,* $q := BB'$*, and* $r := CC'$ *are perspective from a plane* Z *(i.e., the lines* p*,* q*, and* r *lie in the common plane* Z*).*

The principle of duality in *space* affords interchanging of 'plane' and 'point'. The notion 'line' interchanges with itself. The necessary alterations in space are shown on the left side in Table 12. Since the Theorems 5 to 8 are true in space too, Theorem 5 and 7, as well as Theorem 6 and 8, are revealed to be dual to each other in space.

We will now define the principle of duality in geometric algebra $\mathcal{G}_n$ using the completely dual approach of section 3. In a natural way one is led to distinguish between a major and a minor form of the principle of duality depending on whether the sign of the plus-minus notation is changed or not.

Definition 5 *The* major dual S' *of any equation, theorem, or definition* S *in* $\mathcal{G}_n^+$ *or* $\mathcal{G}_n^-$ *is obtained by interchanging juxtaposition with* $*$*,* $\wedge$ *with* $\vee$*,* $\cdot$ *with* $\circ$*, and by reversing the sign of the plus-minus notation. Then* S' *is also an equation, theorem, or definition in* $\mathcal{G}_n^-$ *or* $\mathcal{G}_n^+$*, respectively.*

Theorem 9 The Major Principle of Duality
Any statement S*, deduced in the frame of the geometric algebra* $\mathcal{G}_n^+$*, is true if and only if the major dual statement* S' *is true in the geometric algebra* $\mathcal{G}_n^-$*.*

Proof. The isomorphism (25) guarantees the simultaneous validity of statement S and S'. □

Definition 6 *The* minor dual S' *of any equation, theorem, or definition* S *in* $\mathcal{G}_n^+$ *or* $\mathcal{G}_n^-$ *is obtained by interchanging juxtaposition with* $*$*,* $\wedge$ *with* $\vee$*,* $\cdot$ *with* $\circ$*, and by replacing the grades of a multivector with the corresponding dual grades* $(k \to n-k)$*. Then* S' *is also an equation, theorem, or definition in* $\mathcal{G}_n^+$ *or* $\mathcal{G}_n^-$*, respectively.*

Theorem 10 The Minor Principle of Duality
Any statement S*, deduced in the frame of the geometric algebra* $\mathcal{G}_n^+$ *or* $\mathcal{G}_n^-$*, is true if and only if the minor dual statement* S' *is true in the geometric algebra* $\mathcal{G}_n^+$ *or* $\mathcal{G}_n^-$*, respectively.*

Proof. The isomorphisms (25) and (20) guarantee the simultaneous validity of statement S and S'. □

The Tables 10, 11 and 12 show that the principle of duality in synthetic projective geometry (in its major form) corresponds one-to-one to the major principle of duality in geometric algebra.

To close this section we apply the principle of duality to Desargues' theorem formulated in geometric algebra. The Theorems 11 to 14 provide the conditions for three points in the plane $\mathcal{G}_3$ to be collinear, and three lines in the plane $\mathcal{G}_3$ to be concurrent.

Theorem 11 *The following statements are equivalent:*

i) *The points $P_{\bar{1}}^{+}$, $Q_{\bar{1}}^{+}$, $R_{\bar{1}}^{+}$ are collinear, i.e., they lie on a common line.*

ii) $(P_{\bar{1}}^{+} \wedge Q_{\bar{1}}^{+}) \vee R_{\bar{1}}^{+} = 0$.

iii) $\langle P_{\bar{1}}^{+} * Q_{\bar{1}}^{+} * R_{\bar{1}}^{+} \rangle_3^{+} = 0$.

Proof.

i) ⇔ ii) From $0 = (P_{\bar{1}}^{+} \wedge Q_{\bar{1}}^{+}) \vee R_{\bar{1}}^{+} = [(P_{\bar{1}}^{+} \wedge Q_{\bar{1}}^{+})^{\sim} \wedge (R_{\bar{1}}^{+})^{\sim}]^{\sim} = (\mathbb{I}^{+})^2 \langle (P_{\bar{1}}^{+} \wedge Q_{\bar{1}}^{+}) R_{\bar{1}}^{+} \rangle_3^{+} (\mathbb{I}^{+})^{-1} = (\mathbb{I}^{+})^2 [(P_{\bar{1}}^{+} \wedge Q_{\bar{1}}^{+}) \wedge R_{\bar{1}}^{+}]^{\sim} \Leftrightarrow (P_{\bar{1}}^{+} \wedge Q_{\bar{1}}^{+}) \wedge R_{\bar{1}}^{+} = 0$ follows $(P_{\bar{1}}^{+} \wedge Q_{\bar{1}}^{+}) \vee R_{\bar{1}}^{+} = 0 \Leftrightarrow (P_{\bar{1}}^{+} \wedge Q_{\bar{1}}^{+}) \vee R_{\bar{1}}^{+} = 0$ and $(P_{\bar{1}}^{+} \wedge Q_{\bar{1}}^{+}) \wedge R_{\bar{1}}^{+} = 0 \Leftrightarrow$ The points $P_{\bar{1}}^{+}$, $Q_{\bar{1}}^{+}$, $R_{\bar{1}}^{+}$ are collinear.

ii) ⇔ iii) $0 = (P_{\bar{1}}^{+} \wedge Q_{\bar{1}}^{+}) \vee R_{\bar{1}}^{+} = \langle (P_{\bar{1}}^{+} \wedge Q_{\bar{1}}^{+}) * R_{\bar{1}}^{+} \rangle_3^{-} = \langle [(P_{\bar{1}}^{+})^{\sim} \vee (Q_{\bar{1}}^{+})^{\sim}]^{\sim} * R_{\bar{1}}^{+} \rangle_3^{-} = \langle \langle P_{\bar{1}}^{+} * (\mathbb{I}^{-})^{-1} * Q_{\bar{1}}^{+} * (\mathbb{I}^{-})^{-1} \rangle_2^{-} * (\mathbb{I}^{-})^{-1} * Q_{\bar{1}}^{+} \rangle_3^{-} = (\mathbb{I}^{-})^2 \langle P_{\bar{1}}^{+} * Q_{\bar{1}}^{+} * R_{\bar{1}}^{+} \rangle^{-} * (\mathbb{I}^{-})^{-1} \Leftrightarrow \langle P_{\bar{1}}^{+} * Q_{\bar{1}}^{+} * R_{\bar{1}}^{+} \rangle_3^{+} = 0$.

□

Theorem 12 *The following statements are equivalent:*

i) *The lines $p_{\bar{2}}^{+}$, $q_{\bar{2}}^{+}$, $r_{\bar{2}}^{+}$ are concurrent, i.e., they pass through a common point.*

ii) $(p_{\bar{2}}^{+} \vee q_{\bar{2}}^{+}) \wedge r_{\bar{2}}^{+} = 0$.

iii) $\langle p_{\bar{2}}^{+} q_{\bar{2}}^{+} r_{\bar{2}}^{+} \rangle^{+} = 0$.

Proof. Apply minor duality to Theorem 11. □

Theorem 13 *The following statements are equivalent:*

i) *The lines $P_{\bar{1}}^{-}$, $Q_{\bar{1}}^{-}$, $R_{\bar{1}}^{-}$ are concurrent, i.e., they pass through a common point.*

ii) $(P_{\bar{1}}^{-} \vee Q_{\bar{1}}^{-}) \wedge R_{\bar{1}}^{-} = 0$.

iii) $\langle P_{\bar{1}}^{-} Q_{\bar{1}}^{-} R_{\bar{1}}^{-} \rangle_3^{-} = 0$.

Proof. Apply major duality to Theorem 11. □

Theorem 14 *The following statements are equivalent:*

i) *The points $p_{\bar{2}}^{-}$, $q_{\bar{2}}^{-}$, $r_{\bar{2}}^{-}$ are collinear, i.e., they lie in a common line.*

ii) $(p_{\bar{2}}^- \wedge q_{\bar{2}}^-) \vee r_{\bar{2}}^- = 0$.

iii) $\langle p_{\bar{2}}^- * q_{\bar{2}}^- * r_{\bar{2}}^- \rangle^- = 0$.

Proof. Apply major duality to Theorem 12. □

Let $\Delta = \{A_{\bar{1}}^+, B_{\bar{1}}^+, C_{\bar{1}}^+\}$ and $\Delta' = \{A_{\bar{1}}'^+, B_{\bar{1}}'^+, C_{\bar{1}}'^+\}$ be two generic triangles in the field $\mathcal{G}_3^+$ with

$$\begin{array}{ll} J^+ = \langle J^+ \rangle_3 & J'^+ = \langle J'^+ \rangle_3 \\ \quad := A_{\bar{1}}^+ \wedge B_{\bar{1}}^+ \wedge C_{\bar{1}}^+ & \quad := A_{\bar{1}}'^+ \wedge B_{\bar{1}}'^+ \wedge C_{\bar{1}}'^+ \\ \quad =: \mu(\mathbb{I}^+)^{-1}, & \quad =: \mu'(\mathbb{I}^+)^{-1}, \end{array} \tag{71}$$

$$\begin{array}{rcl} (J^+)^2 & = & -(A_{\bar{1}}^+)^2 (B_{\bar{1}}^+)^2 (C_{\bar{1}}^+)^2, \\ (J'^+)^2 & = & -(A_{\bar{1}}'^+)^2 (B_{\bar{1}}'^+)^2 (C_{\bar{1}}'^+)^2. \end{array} \tag{72}$$

The corresponding sides,

$$\begin{array}{lll} a_{\bar{2}}^+ := B_{\bar{1}}^+ \wedge C_{\bar{1}}^+, & b_{\bar{2}}^+ := C_{\bar{1}}^+ \wedge A_{\bar{1}}^+, & c_{\bar{2}}^+ := A_{\bar{1}}^+ \wedge B_{\bar{1}}^+, \\ a_{\bar{2}}'^+ := B_{\bar{1}}'^+ \wedge C_{\bar{1}}'^+, & b_{\bar{2}}'^+ := C_{\bar{1}}'^+ \wedge A_{\bar{1}}'^+, & c_{\bar{2}}'^+ := A_{\bar{1}}'^+ \wedge B_{\bar{1}}'^+, \end{array} \tag{73}$$

the connecting lines between homologous angles, and the intersecting points between homologous sides,

$$\begin{array}{lll} p_{\bar{2}}^+ := A_{\bar{1}}^+ \wedge A_{\bar{1}}'^+, & q_{\bar{2}}^+ := B_{\bar{1}}^+ \wedge B_{\bar{1}}'^+, & r_{\bar{2}}^+ := C_{\bar{1}}^+ \wedge C_{\bar{1}}'^+, \\ P_{\bar{1}}^+ := a_{\bar{2}}^+ \vee a_{\bar{2}}'^+, & Q_{\bar{1}}^+ := b_{\bar{2}}^+ \vee b_{\bar{2}}'^+, & R_{\bar{1}}^+ := c_{\bar{2}}^+ \vee c_{\bar{2}}'^+, \end{array} \tag{74}$$

transform under multiplication from the right side with the pseudoscalar J^+ or J'^+ into the following homogeneous multivectors.

$$\begin{array}{rcl} a_{\bar{2}}^+ J^+ & = & -(B_{\bar{1}}^+ \wedge C_{\bar{1}}^+)(C_{\bar{1}}^+ \wedge B_{\bar{1}}^+ \wedge A_{\bar{1}}^+) \\ & = & -\langle B_{\bar{1}}^+ C_{\bar{1}}^+ \rangle_2^+ \langle C_{\bar{1}}^+ B_{\bar{1}}^+ A_{\bar{1}}^+ \rangle_3^+ \\ & = & -(B_{\bar{1}}^+)^2 (C_{\bar{1}}^+)^2 A_{\bar{1}}^+ \\ b_{\bar{2}}^+ J^+ & = & -(A_{\bar{1}}^+)^2 (C_{\bar{1}}^+)^2 B_{\bar{1}}^+ \\ c_{\bar{2}}^+ J^+ & = & -(A_{\bar{1}}^+)^2 (B_{\bar{1}}^+)^2 C_{\bar{1}}^+ \end{array} \tag{75a}$$

$$\begin{array}{rcl} a_{\bar{2}}'^+ J'^+ & = & -(B_{\bar{1}}'^+)^2 (C_{\bar{1}}'^+)^2 A_{\bar{1}}'^+ \\ b_{\bar{2}}'^+ J'^+ & = & -(A_{\bar{1}}'^+)^2 (C_{\bar{1}}'^+)^2 B_{\bar{1}}'^+ \\ c_{\bar{2}}'^+ J'^+ & = & -(A_{\bar{1}}'^+)^2 (B_{\bar{1}}'^+)^2 C_{\bar{1}}'^+ \end{array} \tag{75b}$$

$$
\begin{aligned}
P_{\bar{1}}^{+} J^{+} &= \frac{(\mathbb{I}^{+})^2}{\mu'} \langle a_{\bar{2}}^{+} J^{+} a_{\bar{2}}'^{+} J'^{+} \rangle_{2}^{+} \\
&= \frac{(\mathbb{I}^{+})^2}{\mu'} (B_{\bar{1}}^{+})^2 (C_{\bar{1}}^{+})^2 (B_{\bar{1}}'^{+})^2 (C_{\bar{1}}'^{+})^2 \langle A_{\bar{1}}^{+} A_{\bar{1}}'^{+} \rangle_{2}^{+} \\
&= \frac{(\mathbb{I}^{+})^2}{\mu'} (B_{\bar{1}}^{+})^2 (C_{\bar{1}}^{+})^2 (B_{\bar{1}}'^{+})^2 (C_{\bar{1}}'^{+})^2 p_{\bar{2}}^{+} \qquad (76) \\
Q_{\bar{1}}^{+} J^{+} &= \frac{(\mathbb{I}^{+})^2}{\mu'} (C_{\bar{1}}^{+})^2 (A_{\bar{1}}^{+})^2 (C_{\bar{1}}'^{+})^2 (A_{\bar{1}}'^{+})^2 q_{\bar{2}}^{+} \\
R_{\bar{1}}^{+} J^{+} &= \frac{(\mathbb{I}^{+})^2}{\mu'} (A_{\bar{1}}^{+})^2 (B_{\bar{1}}^{+})^2 (A_{\bar{1}}'^{+})^2 (B_{\bar{1}}'^{+})^2 r_{\bar{2}}^{+}
\end{aligned}
$$

We can now state the Desargues' theorem in the plane $\mathcal{G}_3$.

Theorem 15 $\langle P_{\bar{1}}^{+} * Q_{\bar{1}}^{+} * R_{\bar{1}}^{+} \rangle_{3}^{+} = 0 \Leftrightarrow \langle p_{\bar{2}}^{+} q_{\bar{2}}^{+} r_{\bar{2}}^{+} \rangle^{+} = 0.$

Proof.

$$
\begin{aligned}
\langle P_{\bar{1}}^{+} * Q_{\bar{1}}^{+} * R_{\bar{1}}^{+} \rangle_{3}^{+} &= \frac{(\mathbb{I}^{+})^2}{\mu^3} \langle P_{\bar{1}}^{+} J^{+} Q_{\bar{1}}^{+} J^{+} R_{\bar{1}}^{+} J^{+} \rangle^{+} (\mathbb{I}^{+})^{-1} \\
&= \frac{(J^{+} J'^{+})^2}{\mu^3 \mu'^3} \langle p_{\bar{2}}^{+} q_{\bar{2}}^{+} r_{\bar{2}}^{+} \rangle^{+} (\mathbb{I}^{+})^{-1} \\
&= \frac{(\mathbb{I}^{+})^2}{J^{+} J'^{+}} \langle p_{\bar{2}}^{+} q_{\bar{2}}^{+} r_{\bar{2}}^{+} \rangle^{+} (\mathbb{I}^{+})^{-1}
\end{aligned}
$$

□

Theorem 16 $\langle P_{\bar{1}}^{-} Q_{\bar{1}}^{-} R_{\bar{1}}^{-} \rangle_{3}^{-} = 0 \Leftrightarrow \langle p_{\bar{2}}^{-} * q_{\bar{2}}^{-} * r_{\bar{2}}^{-} \rangle^{-} = 0.$

Proof. Apply major duality to Theorem 15. □

Theorem 15 and 16 are clearly self-dual and may also represent Desargues' theorem in the bundle $\mathcal{G}_3$.

For the proof of Desargues' theorem in *space* $\mathcal{G}_4$, we start with two triangles $\Delta = \{A_{\bar{1}}^{+}, B_{\bar{1}}^{+}, C_{\bar{1}}^{+}\}$ and $\Delta' = \{A'^{+}_{\bar{1}}, B'^{+}_{\bar{1}}, C'^{+}_{\bar{1}}\}$ that lie perspective from a center $Z_{\bar{1}}^{+}$. Thus we may write,

$$
\begin{aligned}
\alpha' A'^{+}_{\bar{1}} &= Z_{\bar{1}}^{+} + \alpha A_{\bar{1}}^{+}, \\
\beta' B'^{+}_{\bar{1}} &= Z_{\bar{1}}^{+} + \beta B_{\bar{1}}^{+}, \qquad (77) \\
\gamma' C'^{+}_{\bar{1}} &= Z_{\bar{1}}^{+} + \gamma C_{\bar{1}}^{+},
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha'\beta'\gamma' A'^{+}_{\bar{1}} \wedge B'^{+}_{\bar{1}} \wedge C'^{+}_{\bar{1}} &= Z_{\bar{1}}^{+} \wedge (\beta\gamma B_{\bar{1}}^{+} \wedge C_{\bar{1}}^{+} + \alpha\gamma C_{\bar{1}}^{+} \wedge A_{\bar{1}}^{+} + \qquad (78) \\
&\quad + \alpha\beta A_{\bar{1}}^{+} \wedge B_{\bar{1}}^{+}) + \alpha\beta\gamma A_{\bar{1}}^{+} \wedge B_{\bar{1}}^{+} \wedge C_{\bar{1}}^{+}.
\end{aligned}
$$

Hence the 2-vector

$$z_{\bar{2}}^{+} := \beta\gamma B_{\bar{1}}^{+} \wedge C_{\bar{1}}^{+} + \alpha\gamma C_{\bar{1}}^{+} \wedge A_{\bar{1}}^{+} + \alpha\beta A_{\bar{1}}^{+} \wedge B_{\bar{1}}^{+} \tag{79}$$

represents the line of intersection of the plane $A'^{+}_{\bar{1}} \wedge B'^{+}_{\bar{1}} \wedge C'^{+}_{\bar{1}}$ and the plane $A_{\bar{1}}^{+} \wedge B_{\bar{1}}^{+} \wedge C_{\bar{1}}^{+}$, if the two planes are different. In any case $z_{\bar{2}}^{+}$ is the common line of

$$\begin{aligned} P_{\bar{1}}^{+} &:= \gamma C_{\bar{1}}^{+} - \beta B_{\bar{1}}^{+}, \\ Q_{\bar{1}}^{+} &:= \alpha A_{\bar{1}}^{+} - \gamma C_{\bar{1}}^{+}, \\ R_{\bar{1}}^{+} &:= \beta B_{\bar{1}}^{+} - \alpha A_{\bar{1}}^{+}, \end{aligned} \tag{80}$$

whose expressions can be deduced from the equations

$$\begin{aligned} \alpha'\beta' A'^{+}_{\bar{1}} \wedge B'^{+}_{\bar{1}} &= Z_{\bar{1}}^{+} \wedge (\beta B_{\bar{1}}^{+} - \alpha A_{\bar{1}}^{+}) + \alpha\beta A_{\bar{1}}^{+} \wedge B_{\bar{1}}^{+}, \\ \beta'\gamma' B'^{+}_{\bar{1}} \wedge C'^{+}_{\bar{1}} &= Z_{\bar{1}}^{+} \wedge (\gamma C_{\bar{1}}^{+} - \beta B_{\bar{1}}^{+}) + \beta\gamma B_{\bar{1}}^{+} \wedge C_{\bar{1}}^{+}, \\ \gamma'\alpha' C'^{+}_{\bar{1}} \wedge A'^{+}_{\bar{1}} &= Z_{\bar{1}}^{+} \wedge (\alpha A_{\bar{1}}^{+} - \gamma C_{\bar{1}}^{+}) + \gamma\alpha C_{\bar{1}}^{+} \wedge A_{\bar{1}}^{+}. \end{aligned} \tag{81}$$

Direct computation shows:

$$\begin{aligned} z_{\bar{2}}^{+} \wedge P_{\bar{1}}^{+} = 0 \quad &\text{and} \quad z_{\bar{2}}^{+} \vee P_{\bar{1}}^{+} = 0, \\ z_{\bar{2}}^{+} \wedge Q_{\bar{1}}^{+} = 0 \quad &\text{and} \quad z_{\bar{2}}^{+} \vee Q_{\bar{1}}^{+} = 0, \\ z_{\bar{2}}^{+} \wedge R_{\bar{1}}^{+} = 0 \quad &\text{and} \quad z_{\bar{2}}^{+} \vee R_{\bar{1}}^{+} = 0. \end{aligned} \tag{82}$$

For the other direction of Desargues' theorem in space we start with two triangles Δ and Δ' where the intersecting points $P_{\bar{1}}^{+}$, $Q_{\bar{1}}^{+}$, and $R_{\bar{1}}^{+}$ lie on a common line $z_{\bar{2}}^{+}$. Hence we can write:

$$\begin{aligned} \beta B_{\bar{1}}^{+} &= \alpha A_{\bar{1}}^{+} + R_{\bar{1}}^{+}, & \beta' B'^{+}_{\bar{1}} &= \alpha' A'^{+}_{\bar{1}} + R_{\bar{1}}^{+}, \\ \alpha A_{\bar{1}}^{+} &= \gamma C_{\bar{1}}^{+} + Q_{\bar{1}}^{+}, & \alpha' A'^{+}_{\bar{1}} &= \gamma' C'^{+}_{\bar{1}} + Q_{\bar{1}}^{+}, \\ \gamma C_{\bar{1}}^{+} &= \beta B_{\bar{1}}^{+} + P_{\bar{1}}^{+}, & \gamma' C'^{+}_{\bar{1}} &= \beta' B'^{+}_{\bar{1}} + P_{\bar{1}}^{+}. \end{aligned} \tag{83}$$

The 1-vector

$$Z_{\bar{1}}^{+} := \alpha' A'^{+}_{\bar{1}} - \alpha A_{\bar{1}}^{+} = \beta' B'^{+}_{\bar{1}} - \beta B_{\bar{1}}^{+} = \gamma' C'^{+}_{\bar{1}} - \gamma C_{\bar{1}}^{+} \tag{84}$$

clearly represents the common intersection point of the lines $A_{\bar{1}}^{+} \wedge A'^{+}_{\bar{1}}$, $B_{\bar{1}}^{+} \wedge B'^{+}_{\bar{1}}$, and $C_{\bar{1}}^{+} \wedge C'^{+}_{\bar{1}}$, i.e., it is the center of perspectivity of the triangles Δ and Δ'. Thus, we have proven Desargues' theorem in space.

By major duality in space, we find that two 'triplanes' $\Delta = \{A_{\bar{1}}^{-}, B_{\bar{1}}^{-}, C_{\bar{1}}^{-}\}$ and $\Delta' = \{A'^{-}_{\bar{1}}, B'^{-}_{\bar{1}}, C'^{-}_{\bar{1}}\}$ are perspective from a plane $Z_{\bar{1}}^{-}$, i.e., the lines of intersection $p_{\bar{2}}^{-}$, $q_{\bar{2}}^{-}$, and $r_{\bar{2}}^{-}$ lie in the common plane $Z_{\bar{1}}^{-}$, if and only if the planes of connection $P_{\bar{1}}^{-}$, $Q_{\bar{1}}^{-}$, and $R_{\bar{1}}^{-}$ pass through a common line $z_{\bar{2}}^{-}$.

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